PERGAMON

Pattern Recognition 32 (1999) 947–971

PATTERN RECOGNITION
THE JOURNAL OF THE PATTERN RECOGNITION SOCIETY

Regulated morphological operations[1]

Gady Agam*[,2], Its'hak Dinstein

Department of Electrical and Computer Engineering, Ben-Gurion University of the Negev, Beer-Sheva 84105, Israel

Received 5 June 1997; received in revised form 21 July 1998

Abstract

In this paper regulated morphological operations are defined by extending the fitting interpretation of the ordinary morphological operations. The defined operations have a controllable strictness, and so they are less sensitive to noise and small intrusions or protrusions on the boundaries of shapes. The properties of the defined operations are described, and the relations between them and some other non-linear operations are discussed. Given an existing morphological algorithm, it is possible to try and improve the results obtained by it by using the regulated operations instead of the ordinary operations with strictness that may be optimized according to some optimization criteria. Several examples of the proposed approach are presented. © 1999 Pattern Recognition Society. Published by Elsevier Science Ltd. All rights reserved.

Keywords: Image processing; Mathematical morphology; Fuzzy morphology; Soft morphology; Regulated morphology; Document analysis

1. Introduction

Considering the fitting interpretation of the binary morphological erosion and dilation operations, it is possible to observe that they are based on opposing strict approaches. The binary dilation collects shifts for which the kernel set intersects the object set without taking into account what is the size of the intersection, whereas the binary erosion collects shifts for which the kernel set is completely contained within the object set without considering shifts for which some kernel elements are not contained within the object set. As a result of these strict approaches, the ordinary morphological operations are sensitive to noise and small intrusions or protrusions on the boundary of shapes. In order to solve this problem, various extensions to the ordinary morphological operations have been proposed. These extensions could be classified into two major groups: fuzzy morphological operations [1, 2], and soft morphological operations [3, 4].

The fuzzy morphological operations extend the ordinary morphological operations by using fuzzy sets, where for fuzzy sets the union operation is replaced by a maximum operation, and the intersection operation is replaced by a minimum operation. When using a fuzzy kernel set it is possible to give more weight to some elements in it, and so to reduce the sensitivity to an incomplete fit of the kernel set to the object set. It should be noted that even though the kernel is fuzzy, the fuzzy dilation and erosion still take extreme approaches. The fuzzy dilation uses a maximum operation, whereas the fuzzy erosion uses a minimum operation. The soft morphological operations use a structuring system which is

*Corresponding author.

[1]This work was partially supported by The Paul Ivanier Center for Robotics and Production Automation, Ben-Gurion University of the Negev, Beer-Sheva 84105, Israel.

[2]Current address: Ecole Polytechique, Department of Electrical and Computer Engineering, Montreal H3C 3A7, Canada.

PII: S0031-3203(98)00120-4

composed of two kernel sets: hard and soft, and an order parameter. By determining the hard and soft kernel sets, and the order parameter it is possible to control the morphological operations, and thereby reduce their sensitivity to noise. While the soft erosion and dilation manage to maintain major properties of ordinary erosion and dilation, the soft open and close are not idempotent in general. The soft open and close are proven to be idempotent only in a few cases of special kernel sets when using special order parameters. As demonstrated later in this paper, it is possible to compose the soft morphological operations by using a combination of ordinary and regulated morphological operations.

The relations between order-statistic filters [5] and ordinary morphological operations are described in Ref. [6], where it is shown that the ordinary morphological operations can be obtained as special cases of order-statistic filters. Based on this observation it is suggested in Refs. [7, 8] to reduce the sensitivity of the ordinary morphological operations to noise by using modified morphological operations that are based on order-statistic filters. This paper defines regulated morphological operations, and shows how the fitting property of the ordinary morphological operations is controlled in these operations. The defined regulated morphological operations include: regulated erosion, regulated dilation, regulated open, and regulated close. The properties of the regulated morphological operations are discussed and it is shown that they possess many of the properties of the ordinary morphological operations. In particular, it is shown that the regulated open and close are idempotent for an arbitrary kernel and strictness parameter. Since the regulated morphological operations possess many of the properties of the ordinary morphological operations, it is possible to use the regulated morphological operations in existing algorithms that are based on morphological operations in order to improve their performance, where the strictness parameter of the regulated morphological operations may be optimized according to some criteria.

The following sections discuss the proposed approach in greater detail. Section 2 defines the regulated erosion and dilation operations, studies their properties, and discusses the relations between them. Section 3 determines the relations between the regulated morphological operations and other operations. Section 4 defines and studies the properties of compound regulated morphological operations. Some examples of regulating other morphological operations based on the basic regulated morphological operations are presented in Section 5. The summary in Section 6 concludes the paper.

2. Basic regulated morphological operations

This section defines regulated morphological operations that have a controllable strictness. By using the strictness parameter of the operations it is possible to control the sensitivity of the operations to noise and small intrusions or protrusions on the boundary of shapes, and thereby prevent excessive dilation or erosion. The properties of the regulated morphological operations are discussed, and it is shown that the ordinary morphological operations may be obtained as a special case of the regulated operations. Finally the relations between the regulated erosion and dilation are discussed.

2.1. Regulated dilation

Given two sets $A, B \subset \mathbb{Z}^N$, the morphological dilation of A by B is defined [9] by

$$A \oplus B \equiv \{x \mid \exists a \in A,\, b \in B : x = a + b\} = \bigcup_{a \in A} (B)_a \tag{1}$$

where $(B)_a$ is a shift of B by a defined by: $(B)_a \equiv \{x \mid \exists b \in B: x = a + b\}$. When A is a set of binary image pixels, and B is a set of binary kernel pixels, the dilation of A by B results in a dilation of the shapes in A (provided that the origin pixel belongs to the kernel B). The dilation operation may be interpreted in various ways. In particular, the dilation of A by B may be obtained as the union of all the possible shifts for which the reflected and shifted B intersects A. That is

$$A \oplus B = \{x \mid (A \cap (\breve{B})_x) \neq \emptyset\} \tag{2}$$

where $\breve{B}$ is the reflection of B given by $\breve{B} \equiv \{x \mid \exists b \in B: x = -b\}$. As could be noted, the reflection of B in $\mathbb{Z}^2$ is equivalent to its rotation by 180°. By using the fitting interpretation of dilation (2), the morphological dilation of A by B can be extended by combining the size of the intersection into the dilation process. In that sense, a given shift is included in the dilation of A only if the intersection between A and the reflected and shifted B is big enough. The obtained advantage of the regulated dilation is the prevention of excessive dilation caused by small intersections with the object set.

Definition 1. The regulated dilation of A by B with a strictness of s is defined by:

$$A \overset{s}{\oplus} B \equiv \{x \mid \#(A \cap (\breve{B})_x) \geqslant s\};$$
$$s \in [1, \min(\#A, \#B)] \tag{3}$$

where the symbol $\#$ denotes the cardinality of a set.

It should be noted that since $\#(A \cap (\breve{B})_x) \leqslant \min(\#A, \#B)$ for every x, the strictness s is bounded by $\min(\#A, \#B)$.

Fig. 1 demonstrates the fitting interpretation of the ordinary and the regulated dilation. Figs 1a and b presents the original shape and the kernel set respectively, where the black square indicates the origin of the kernel.

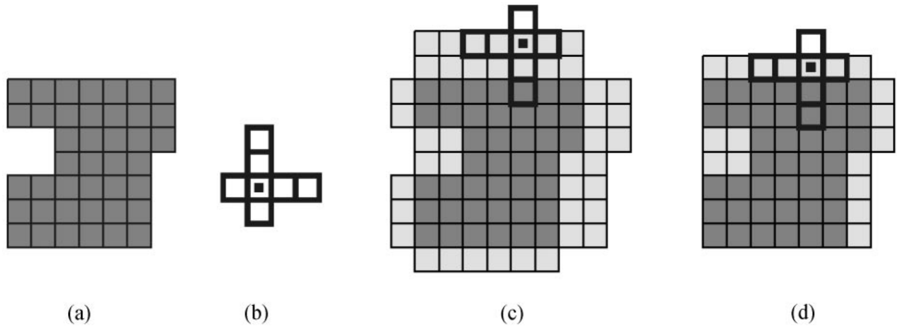


Fig. 1. The fitting interpretation of the ordinary and the regulated dilation. (a) The original shape, (b) the kernel set, (c) the result of an ordinary dilation obtained as the union of all the possible shifts of the reflected kernel for which the intersection with the original shape is not empty, (d) the result of a regulated dilation with a strictness of two, obtained as the union of all the possible shifts of the reflected kernel for which the size of the intersection with the original shape is greater than or equal to two. The elements that were added by the dilation operations are marked in light gray. As can be observed, the result obtained by the regulated dilation is smaller than the result obtained by the ordinary dilation.

Fig. 1c presents the result of an ordinary dilation obtained as the union of all the possible shifts of the reflected kernel for which the size of the intersection with the original shape is greater or equal to one. Fig. 1d presents the result of a regulated dilation with a strictness of two, obtained as the union of all the possible shifts of the reflected kernel for which the size of the intersection with the original shape is greater or equal to two. The elements that were added by the dilation are marked in these figures in light gray. As can be observed, the result obtained by the regulated dilation is smaller than the result obtained by the ordinary dilation, since in the regulated dilation the kernel has to penetrate deeper into the shape in order to add an element to the dilated shape.

Proposition 2. *The regulated dilation is decreasing with respect to the strictness* s:

$$A \overset{s1}{\oplus} B \subseteq A \overset{s2}{\oplus} B \Leftrightarrow s1 \geqslant s2 \tag{4}$$

When s *is minimal, the regulated dilation results in the ordinary dilation*:

$$A \overset{1}{\oplus} B = A \oplus B \tag{5}$$

Proof. We only prove the first part of the proposition. The second part results directly from the definition of the regulated dilation. Assume first that $s1 \geqslant s2$. Therefore, by using the definition of the regulated dilation, we get that $\#(A \cap (\breve{B})_x) \geqslant s1 \geqslant s2$ for all $x \in A \overset{s1}{\oplus} B$, and so $x \in A \overset{s2}{\oplus} B$. However, there may exist $x \in A \overset{s2}{\oplus} B$ such that $s1 > \#(A \cap (\breve{B})_x) \geqslant s2$, and so $x \notin A \overset{s1}{\oplus} B$. Assume now that $A \overset{s1}{\oplus} B \subset A \overset{s2}{\oplus} B$. Therefore, there exists $x \in A \overset{s2}{\oplus} B$ such that $x \notin A \overset{s1}{\oplus} B$. Hence, for that x we get that $s2 \leqslant \#(A \cap (\breve{B})_x) < s1$ and so $s2 \leqslant s1$. □

Corollary 3. *The regulated dilation results in a subset of the ordinary dilation*:

$$A \overset{s}{\oplus} B \subseteq A \oplus B \tag{6}$$

Proposition 4. *The regulated dilation is commutative, increasing with respect to the first and the second arguments, and translation invariant*:

$$A \overset{s}{\oplus} B = B \overset{s}{\oplus} A, \tag{7}$$

$$A \subseteq B \Rightarrow A \overset{s}{\oplus} K \subseteq B \overset{s}{\oplus} K, \tag{8}$$

$$B \subseteq D \Rightarrow A \overset{s}{\oplus} B \subseteq A \overset{s}{\oplus} D, \tag{9}$$

$$(A)_x \overset{s}{\oplus} B = (A \overset{s}{\oplus} B)_x, \tag{10}$$

$$A \overset{s}{\oplus} (B)_x = (A \overset{s}{\oplus} B)_x. \tag{11}$$

Proof. The proofs of these properties result directly from the regulated dilation definition. Consider for example the proof of Eq. (7). Based on the definition of the regulated dilation, it is necessary to show that $\#(A \cap (\breve{B})_x) = \#(B \cap (\breve{A})_x)$. By developing the left side of this equation we get $\#(A \cap (\breve{B})_x) = \#\{a \in A \mid \exists b \in B\colon\ a = -b + x\} = \#\{b \in B \mid \exists a \in A\colon\ b = -a + x\} = \#(B \cap (\breve{A})_x)$. □

Proposition 5. *The regulated dilation of a union (intersection) of sets is bigger (smaller) or equal to the union (intersection) of the regulated dilation of the individual sets*:

$$(A \cup B) \overset{s}{\oplus} K \supseteq (A \overset{s}{\oplus} K) \cup (B \overset{s}{\oplus} K), \tag{12}$$

$$(A \cap B) \overset{s}{\oplus} K \subseteq (A \overset{s}{\oplus} K) \cap (B \overset{s}{\oplus} K). \tag{13}$$

Proof. Consider the proof of Eq. (12). Since $A \cup B \supseteq A$ and $A \cup B \supseteq B$, by using Eq. (8) we get that $(A \cup B) \overset{s}{\oplus} K \supseteq A \overset{s}{\oplus} K$ and $(A \cup B) \overset{s}{\oplus} K \supseteq B \overset{s}{\oplus} K$. Therefore, $(A \cup B) \overset{s}{\oplus} K \supseteq (A \overset{s}{\oplus} K) \cup (B \overset{s}{\oplus} K)$. The proof of Eq. (13) may be obtained similarly. □

It should be noted that an equality in Eq. (12) is always obtained when $s = 1$.

2.2. Regulated erosion

Given two sets $A, B \subset \mathbb{Z}^N$, the morphological erosion of A by B is defined [9] by

$$A \ominus B \equiv \{x \mid \forall b \in B \; \exists a \in A\colon x = a - b\} = \bigcap_{b \in B} (A)_{-b} \quad (14)$$

When A is a set of binary image pixels, and B is a set of binary kernel pixels, the erosion of A by B results in an erosion of the shapes in A (provided that the origin pixel belongs to the kernel B). It is possible to show [9] that dilation and erosion are dual, so that the morphological erosion of A by B can be obtained by dilating the complement of A with the reflected B, and then taking the complement of the result. That is

$$A \ominus B = (A^c \oplus \breve{B})^c \quad (15)$$

where A^c denotes the complement of A defined by: $A^c \equiv \{x \mid x \notin A\}$. The erosion operation may be interpreted in various ways. In particular, the erosion of A by B may be obtained as the union of all the possible shifts for which the shifted B is contained completely within A. That is

$$A \ominus B = \{x \mid (A^c \cap (B)_x) = \emptyset\}. \quad (16)$$

By using Eq. (16) the morphological erosion of A by B can be extended by including in the erosion of A shifts for which the intersection between A^c and the shifted B is small enough. The obtained advantage of the regulated erosion is the prevention of excessive erosion caused by small intersections with the background set.

Definition 6. The regulated erosion of A by B with a strictness of s is defined by:

$$A \overset{s}{\ominus} B \equiv \{x \mid \#(A^c \cap (B)_x) < s\}, \quad s \in [1, \#B] \quad (17)$$

where it is assumed that $\#A < \infty$.

It should be noted that since it is assumed that $\#A < \infty$ then $\#(A^c \cap (B)_x) \leqslant \#B$ for every x, and so the strictness s is bounded by $\#B$.

Fig. 2 demonstrates the fitting interpretation of the ordinary and the regulated erosion. Fig. 2a and b presents the original shape and the kernel set respectively, where the black square indicates the origin of the kernel. Fig. 2c presents the result of an ordinary erosion obtained as the union of all the possible shifts of the kernel that are contained completely within the original shape. Fig. 2d presents the result of a regulated erosion with a strictness of two obtained as the union of all the possible shifts of the kernel for which the size of the intersection with the background of the original shape is less than two. The elements that were removed by the erosion are marked in these figures in light gray. As can be observed, the result obtained by the regulated erosion is larger than the result obtained by the ordinary erosion, since in the regulated erosion the kernel could get into more pixels in order to prevent their removal from the eroded shape.

Proposition 7. *The regulated dilation and erosion are dual in the same sense that exists for the ordinary dilation and*

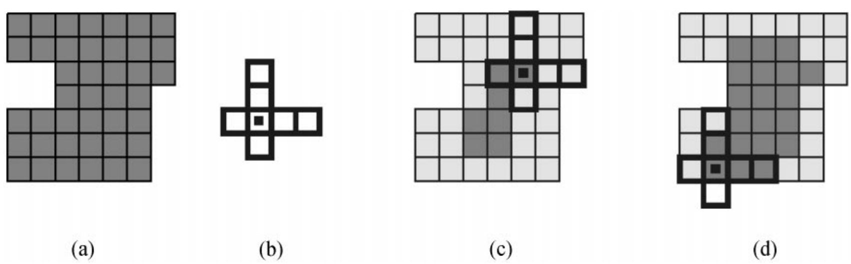


Fig. 2. The fitting interpretation of the ordinary and the regulated erosion: (a) The original shape, (b) the kernel set, (c) the result of an ordinary erosion obtained as the union of all the possible shifts of the kernel that are contained completely within the original shape, (d) the result of a regulated erosion with a strictness of two obtained as the union of all the possible shifts of the kernel for which the size of the intersection with the background of the original shape is less than two. The elements that were removed by the erosion operations are marked in light gray. As can be observed, the result obtained by the regulated erosion is larger than the result obtained by the ordinary erosion.

erosion:

$$A \overset{s}{\ominus} B = (A^c \overset{s}{\oplus} \breve{B})^c \tag{18}$$

Proof. By developing the right side of the proposition according to the regulated dilation definition we get $(A^c \overset{s}{\oplus} \breve{B})^c = \{x \mid \#(A^c \cap (B)_x) \geqslant s\}^c = \{x \mid \#(A^c \cap (B)_x) < s\} = A \overset{s}{\ominus} B$. □

Proposition 8. *The regulated erosion is increasing with respect to the strictness s*:

$$A \overset{s1}{\ominus} B \subseteq A \overset{s2}{\ominus} B \Leftrightarrow s1 \leqslant s2 \tag{19}$$

When s is minimal, the regulated erosion results in the ordinary erosion:

$$A \overset{1}{\ominus} B = A \ominus B \tag{20}$$

Proof. By using Eq. (4) we get that $A^c \overset{s2}{\oplus} \breve{B} \subseteq A^c \overset{s1}{\oplus} \breve{B} \Leftrightarrow s2 \geqslant s1$. Therefore, $(A^c \overset{s2}{\oplus} \breve{B})^c \supseteq (A^c \overset{s1}{\oplus} \breve{B})^c \Leftrightarrow s2 \geqslant s1$, and so by using the duality proposition of the regulated erosion and dilation we get that $A \overset{s2}{\ominus} B \supseteq A \overset{s1}{\ominus} B \Leftrightarrow s2 \geqslant s1$. The proof of the second part results directly from the definition of the regulated erosion. □

Corollary 9. *The regulated erosion results in a superset of the ordinary erosion*:

$$A \overset{s}{\ominus} B \supseteq A \ominus B. \tag{21}$$

Proposition 10. *The regulated erosion is increasing with respect to the first argument, decreasing with respect to the second argument, and translation invariant*:

$$A \subseteq B \Rightarrow A \overset{s}{\ominus} K \subseteq B \overset{s}{\ominus} K, \tag{22}$$

$$B \subseteq D \Rightarrow A \overset{s}{\ominus} B \supseteq A \overset{s}{\ominus} D, \tag{23}$$

$$(A)_x \overset{s}{\ominus} B = (A \overset{s}{\ominus} B)_x, \tag{24}$$

$$A \overset{s}{\ominus} (B)_x = (A \overset{s}{\ominus} B)_{-x}. \tag{25}$$

Proof. The proofs of these properties result directly from the regulated erosion definition. Consider for example the proof of Eq. (22). Since $A \subseteq B \Rightarrow A^c \supseteq B^c$, it follows that $\#(A^c \cap (K)_x) \geqslant \#(B^c \cap (K)_x)$, and so $\{x \mid \#(A^c \cap (K)_x) < s\} \subseteq \{x \mid \#(B^c \cap (K)_x) < s\}$. Therefore, according to the definition of the regulated erosion: $A \overset{s}{\ominus} K \subseteq B \overset{s}{\ominus} K$. □

Proposition 11. *The regulated erosion of a union (intersection) of sets is bigger (smaller) or equal to the union (intersection) of the regulated erosion of the individual sets*:

$$(A \cup B) \overset{s}{\ominus} K \supseteq (A \overset{s}{\ominus} K) \cup (B \overset{s}{\ominus} K), \tag{26}$$

$$(A \cap B) \overset{s}{\ominus} K \subseteq (A \overset{s}{\ominus} K) \cap (B \overset{s}{\ominus} K). \tag{27}$$

Proof. Consider the proof of Eq. (26). Since $A \cup B \supseteq A$ and $A \cup B \supseteq B$, by using Eq. (22) we get that $(A \cup B) \overset{s}{\ominus} K \supseteq A \overset{s}{\ominus} K$ and $(A \cup B) \overset{s}{\ominus} K \supseteq B \overset{s}{\ominus} K$. Therefore, $(A \cup B) \overset{s}{\ominus} K \supseteq (A \overset{s}{\ominus} K) \cup (B \overset{s}{\ominus} K)$. The proof of Eq. (27) may be obtained similarly. □

It should be noted that an equality is always obtained in Eq. (27) when $s = 1$.

2.3. Relations between the regulated erosion and dilation

As stated earlier the regulated erosion and dilation are dual in the same sense that exists for ordinary morphological operations. That is, the regulated erosion may be obtained from the regulated dilation when dilating the complement of the set (the background) with the reflected kernel and then taking the complement of the result. In the rest of this section a more basic relation between the regulated erosion and dilation is developed.

The regulated dilation of A by B may be interpreted as the union of all the possible shifts for which the intersection between A and the reflected and shifted B is big enough. The following proposition states that the same interpretation may be applied to the regulated erosion when using the reflection of the set B and the complement of the strictness s.

Definition 12. The complement of the strictness s relative to the set B is defined by

$$\overline{s_B} \equiv \#B - s + 1. \tag{28}$$

Lemma 13. *The regulated erosion of A by B with a strictness of s may be obtained by*

$$A \overset{s}{\ominus} B = \{x \mid \#(A \cap (B)_x) \geqslant \overline{s_B}\}, \quad s \in [1, \#B]. \tag{29}$$

Proof. By developing the left side of the lemma according to the regulated erosion definition, we get $A \overset{s}{\ominus} B = \{x \mid \#(A^c \cap (B)_x) < s\} = \{x \mid \#(A \cap (B)_x) \geqslant \#B - s + 1\} = \{x \mid \#(A \cap (B)_x) \geqslant \overline{s_B}\}$. □

Proposition 14. *The regulated dilation and erosion may be obtained from each other by reflecting the kernel set and* *complementing the strictness relative to the kernel set*:

$$A \overset{s}{\oplus} B = A \overset{\overline{s_B}}{\ominus} \breve{B}, \tag{30}$$

$$A \overset{s}{\ominus} B = A \overset{\overline{s_B}}{\oplus} \breve{B}, \tag{31}$$

where $s \in [1, \#B]$.

Proof. Results directly from Eq. (29) when using the fact that the complement of the strictness $\overline{s_B}$ relative to B is s. □

Corollary 15. *When B is invariant under reflection (that is* $B = \breve{B}$*), the regulated dilation and erosion of A by B give identical results when using a strictness of*: $s = (\#B + 1)/2$.

Following Proposition 14, and the fact that the regulated dilation (erosion) is decreasing (increasing) with respect to the strictness *s*, it is possible to observe that the regulated dilation (erosion) is turned into erosion (dilation) when increasing the strictness *s* (assuming that the kernel is invariant under reflection). That is, a regulated dilation (erosion) which is too strict is turned into erosion (dilation). Therefore, it is possible to conclude that the regulated dilation and erosion operations are essentially the same, and differ only by the degree of strictness and by a reflection of the kernel.

Fig. 3 presents an example of the regulated dilation. Fig. 3a presents the original image, and Fig. 3b–j presents the results obtained by a regulated dilation with a strictness of 1–9, respectively. The kernel used in this example is a 3×3 square with the origin at its center. The light gray elements in the resulting images represent elements that were removed from the original image. Since the kernel in that example is invariant under reflection, the presented results are also the results obtained by a regulated erosion with a strictness of 9–1, respectively. As could be observed, while by using ordinary morphological operations it is possible to obtain only the extreme ends of the sequence in Fig. 3 (Fig. 3b and j), by using regulated morphological operations it is possible to obtain any part of the sequence by selecting the required strictness.

Based on Eqs. (30) and (31), and the fact that the regulated dilation is commutative, it is possible to construct a proposition concerning the exchange between the arguments of a regulated erosion operation.

Proposition 16. *It is possible to exchange the arguments of a regulated erosion operation provided that the arguments are reflected, and that the strictness is updated*:

$$A \overset{s}{\ominus} B = \breve{B} \overset{\#A - \#B + s}{\ominus} \breve{A} \tag{32}$$

where $s \in [1, \#B]$.

Proof. By developing the left side of the proposition according to the regulated duality proposition, and using the fact that the regulated dilation is commutative, we get $A \overset{s}{\ominus} B = A \overset{\overline{s_B}}{\oplus} \breve{B} = \breve{B} \overset{\#B - s + 1}{\oplus} A = \breve{B} \overset{\#A - (\#B - s + 1) + 1}{\oplus} \breve{A} = \breve{B} \overset{\#A - \#B + s}{\oplus} \breve{A}$. □

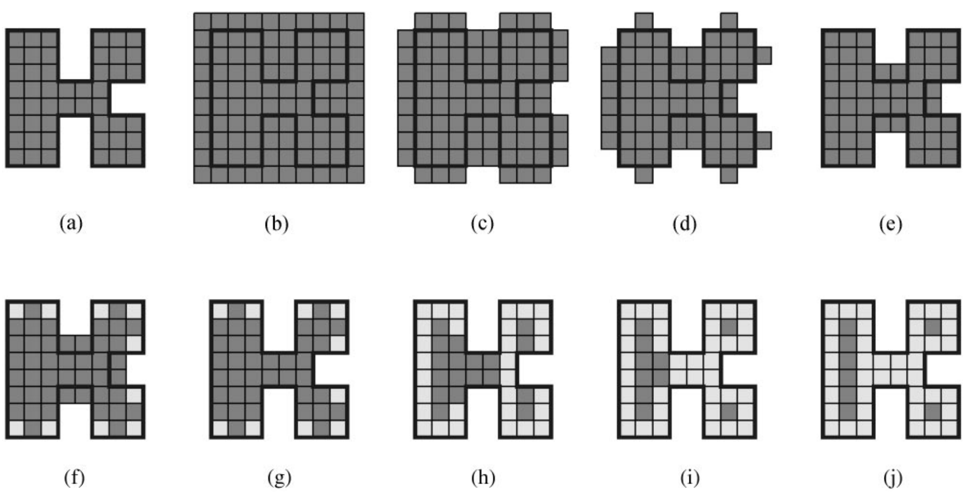


Fig. 3. Demonstration of the regulated dilation and erosion operations. (a) The original shape, (b)–(j) the results obtained by a regulated dilation with a strictness of 1–9, respectively. The kernel that is used in this example is a 3×3 square, with the origin at its center. Since the kernel used in this example is invariant under reflection, these results are identical to those obtained by a regulated erosion with a strictness of 9–1, respectively. The elements that were removed from the original shape are marked in light gray.

2.4. Extensive regulated morphological operations

As was demonstrated, the regulated dilation and erosion operations change their operation from dilation to erosion (or vice versa) when changing their strictness. Therefore, for an arbitrary strictness, the regulated dilation (erosion) is not necessarily extensive (anti-extensive), even if the origin belongs to the kernel set. An extensivity (anti-extensivity) property for dilation (erosion) may be desired when combining the basic regulated operations in order to generate compound operations such as open and close. In this section, an extensive regulated dilation and an anti-extensive regulated erosion operations are defined based on the regulated morphological operations.

Definition 17. The extensive regulated dilation of A by B with a strictness of s is defined by

$$A \underline{\overset{s}{\oplus}} B \equiv (A \overset{s}{\oplus} B) \cup A;\; s \in [1, \min(\#A, \#B)] \tag{33}$$

Since in this definition the result of the regulated dilation is unified with the original set, the defined operation is necessarily extensive (for an arbitrary kernel and strictness).

Proposition 18. *The extensive regulated dilation is decreasing with respect to the strictness* s:

$$A \underline{\overset{s1}{\oplus}} B \subseteq A \underline{\overset{s2}{\oplus}} B \Leftrightarrow s1 \geqslant s2 \tag{34}$$

When s *is minimal, the extensive regulated dilation results in an ordinary dilation*:

$$A \underline{\overset{1}{\oplus}} B = A \oplus B \tag{35}$$

Proof. We only prove the first part of the proposition. The second part results directly from the definition of the extensive regulated dilation. Assume first that $s1 \geqslant s2$. Thus, by using Eq. (4), we get that $(A \overset{s1}{\oplus} B) \subseteq (A \overset{s2}{\oplus} B)$. Therefore, $(A \overset{s1}{\oplus} B) \cup A \subseteq (A \overset{s2}{\oplus} B) \cup A$ and so $A \underline{\overset{s1}{\oplus}} B \subseteq A \underline{\overset{s2}{\oplus}} B$. Assume now that $A \underline{\overset{s1}{\oplus}} B \subseteq A \underline{\overset{s2}{\oplus}} B$. We have to show that it must be that $s1 \geqslant s2$. Assume that $s1 < s2$. Thus, by using Eq. (4), we get that $(A \overset{s1}{\oplus} B) \supseteq (A \overset{s2}{\oplus} B)$. Therefore, $(A \overset{s1}{\oplus} B) \cup A \supseteq (A \overset{s2}{\oplus} B) \cup A$ and so we get $A \underline{\overset{s1}{\oplus}} B \supsetneq A \underline{\overset{s2}{\oplus}} B$, which contradicts the original assumption that $A \underline{\overset{s1}{\oplus}} B \subseteq A \underline{\overset{s2}{\oplus}} B$. Thus, $s1 \geqslant s2$. □

Proposition 19. *The extensive regulated dilation possesses the following properties*:

$$A \subseteq B \Rightarrow A \underline{\overset{s}{\oplus}} K \subseteq B \underline{\overset{s}{\oplus}} K, \tag{36}$$

$$B \subseteq D \Rightarrow A \underline{\overset{s}{\oplus}} B \subseteq A \underline{\overset{s}{\oplus}} D, \tag{37}$$

$$(A)_x \underline{\overset{s}{\oplus}} B = (A \underline{\overset{s}{\oplus}} B)_x, \tag{38}$$

$$(A \cup B) \underline{\overset{s}{\oplus}} K \supseteq (A \underline{\overset{s}{\oplus}} K) \cup (B \underline{\overset{s}{\oplus}} K), \tag{39}$$

$$(A \cap B) \underline{\overset{s}{\oplus}} K \subseteq (A \underline{\overset{s}{\oplus}} K) \cap (B \underline{\overset{s}{\oplus}} K). \tag{40}$$

Proof. The proofs of these properties result directly from the extensive regulated dilation definition, and the respective properties of the regulated dilaiton. □

It should be noted that equality is always obtained in Eq. (39) when $s = 1$.

Definition 20. The anti-extensive regulated erosion of A by B with a strictness of s is defined by

$$A \underline{\overset{s}{\ominus}} B \equiv (A \overset{s}{\ominus} B) \cap A;\; s \in [1, \#B] \tag{41}$$

Since in this definition the result of the regulated erosion is intersected with the original set, the defined operation is necessarily anti-extensive (for an arbitrary kernel and strictness).

Proposition 21. *The extensive regulated dilation and anti-extensive regulated erosion are dual in the same sense that exists for the ordinary dilation and erosion*:

$$A \underline{\overset{s}{\oplus}} B = (A^c \underline{\overset{s}{\ominus}} \breve{B})^c. \tag{42}$$

Proof. The proof of this proposition results directly from the duality of the regulated dilation and erosion, and the definition of the extensive regulated dilation and the anti-extensive regulated erosion. □

Proposition 22. *The anti-extensive regulated erosion is increasing with respect to the strictness* s:

$$A \underline{\overset{s1}{\ominus}} B \subseteq A \underline{\overset{s2}{\ominus}} B \Leftrightarrow s1 \leqslant s2. \tag{43}$$

When s *is minimal, the anti-extensive regulated erosion results in an ordinary erosion*:

$$A \underline{\overset{1}{\ominus}} B = A \ominus B. \tag{44}$$

Proof. By using Eq. (34) we get that $A^c \underline{\overset{s2}{\oplus}} \breve{B} \subseteq A^c \underline{\overset{s1}{\oplus}} \breve{B} \Leftrightarrow s2 \geqslant s1$. Therefore, $(A^c \underline{\overset{s2}{\oplus}} \breve{B})^c \supseteq (A^c \underline{\overset{s1}{\oplus}} \breve{B})^c \Leftrightarrow s2 \geqslant s1$, and so by using the duality proposition of the anti-extensive regulated erosion and the extensive regulated dilation we get that $A \underline{\overset{s2}{\ominus}} B \supseteq A \underline{\overset{s1}{\ominus}} B \Leftrightarrow s2 \geqslant s1$. The proof of the second part of the proposition results directly from the definition of the anti-extensive regulated erosion. □

Proposition 23. *The anti-extensive regulated erosion possesses the following properties*:

$$A \subseteq B \Rightarrow A \underline{\overset{s}{\ominus}} K \subseteq B \underline{\overset{s}{\ominus}} K, \tag{45}$$

$$B \subseteq D \Rightarrow A \underline{\overset{s}{\ominus}} B \supseteq A \underline{\overset{s}{\ominus}} D, \tag{46}$$

$$(A)_x \underline{\overset{s}{\ominus}} B = (A \underline{\overset{s}{\ominus}} B)_x, \tag{47}$$

$$(A \cup B) \underline{\overset{s}{\ominus}} K \supseteq (A \underline{\overset{s}{\ominus}} K) \cup (B \underline{\overset{s}{\ominus}} K), \tag{48}$$

$$(A \cap B) \underline{\overset{s}{\ominus}} K \subseteq (A \underline{\overset{s}{\ominus}} K) \cap (B \underline{\overset{s}{\ominus}} K). \tag{49}$$

Proof. The proofs of these properties result directly from the anti-extensive regulated erosion definition, and the respective properties of the regulated erosion. □

It should be noted that equality is always obtained in Eq. (49) when $s = 1$.

Fig. 4 presents an example of an extensive regulated dilation and an anti-extensive regulated erosion operations in which the original shape and the kernel set are identical to those used in Fig. 3. Figs. 4a–e presents the results of an extensive regulated dilation with a strictness of 1–5, respectively, and Figs. 4f–j presents the results of an anti-extensive regulated erosion with a strictness of 1–5, respectively. The elements that were removed from the original shape are marked in these figures in light gray.

2.5. Examples of the regulated erosion and dilation

This section presents some examples of applications of the regulated erosion and dilation operations, and demonstrates their ability to improve the results obtained by the ordinary operations when using strictness parameter greater than one. Fig. 5 presents the results of an ordinary and a regulated dilation of a simple dashed lines image. The original image is presented in Fig. 5a. The result of an ordinary dilation of the image in Fig. 5a by a line kernel in the direction of 45° is shown in Fig. 5b. The result of a regulated dilation of the same image by the same kernel when using a strictness parameter which is greater than one is demonstrated in Fig. 5c. As can be observed, both the ordinary and regulated dilation operations manage to fill the gaps in the diagonal dashed lines. However, the regulated dilation removes the horizontal dashed lines, whereas the ordinary dilation generates noise due to the dilation of these lines. A complete description of the application of regulated morphological operations for dashed lines reconstruction is presented in details in Ref. [10].

An example of objects boundary smoothing by a regulated dilation is presented in Fig. 6. Fig. 6a presents the initial map image, in which the boundaries of the objects are noisy. Figs. 6b–c demonstrates the results of an ordinary dilation of the image in Fig. 6a by a 2×2 and a 3×3 square kernels, respectively. Fig. 6d shows the result of a regulated dilation of the same image by a 3×3 square kernel when using a strictness parameter of 4. As can be observed, the ordinary dilation operations result in thicker objects with noisy boundaries, whereas the regulated dilation smoothes the boundary of shapes without thickening them. As can be observed, the ordinary dilation operations result in thicker objects with noisy boundaries, whereas the regulated dilation smoothes the boundary of shapes without thickening them.

An example of directional extraction of lines by a regulated dilation operation is presented in Fig. 7. Fig. 7a

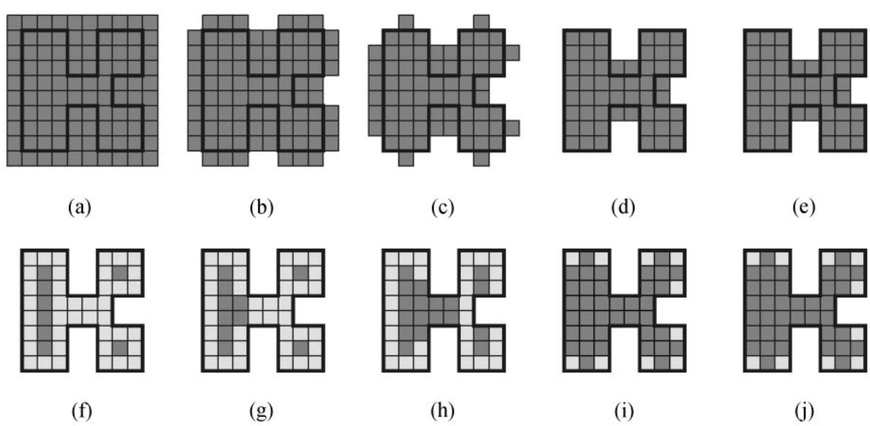
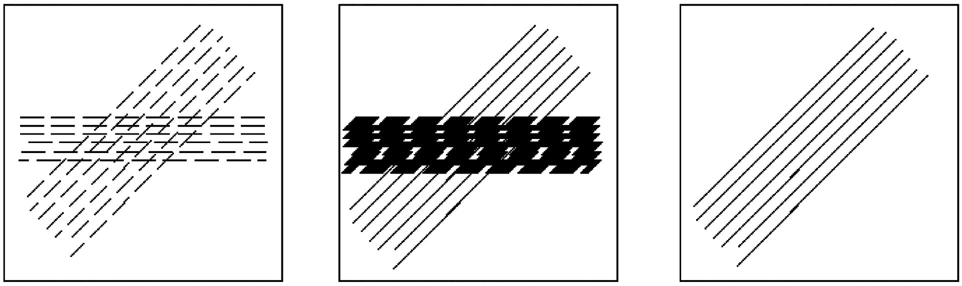
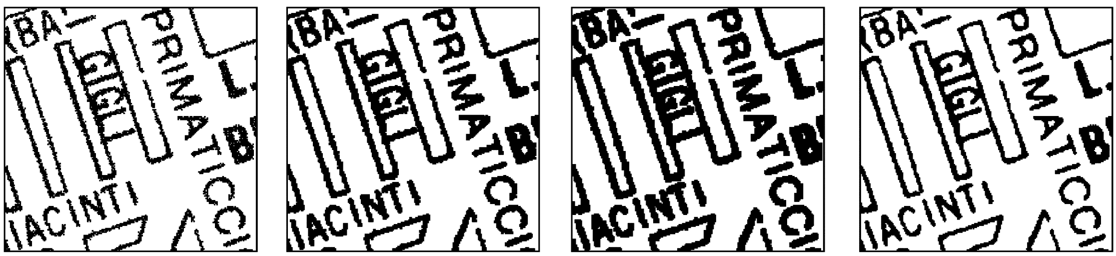


Fig. 4. Demonstration of the extensive regulated dilation and the anti-extensive regulated erosion operations. (a)–(e) The results of an extensive regulated dilation with a strictness of 1–5, respectively, (f)–(j) the results of an anti-extensive regulated erosion with a strictness of 1–5, respectively. The original shape and the kernel set that are used in this example are identical to those used in Fig. 3. The elements that were removed from the original shape are marked in light gray.



(a) (b) (c)

Fig. 5. Demonstration of the ordinary and the regulated dilation of dashed lines. (a) The original image, (b) the result of an ordinary dilation of the image in (a) by a line kernel in the direction of 45°, (c) the result of a regulated dilation of the same image by the same kernel when using a strictness parameter which is greater than one. As can be observed, both the ordinary and the regulated dilation operations manage to fill the gaps in the diagonal dashed lines. However, the regulated dilation removes the horizontal dashed lines, whereas the ordinary dilation generates noise due to the dilation of these lines.



(a) (b) (c) (d)

Fig. 6. Demonstration of objects boundary smoothing by a regulated dilation operation. (a) The initial map image, in which the boundaries of the objects are noisy, (b)–(c) the results of an ordinary dilation of the image in (a) by a 2×2 and a 3×3 square kernels, respectively, (d) the result of a regulated dilation of the same image by a 3×3 square kernel when using a strictness parameter of 4. As can be observed, the ordinary dilation operations result in thicker objects with noisy boundaries, whereas the regulated dilation smoothes the boundary of shapes without thickening them.

(a) (b) (c)

Fig. 7. Demonstration of directional extraction of lines by a regulated dilation operation. (a) The initial map image, containing one side edges (in the lower right direction) of objects in the original image, (b) the result of an ordinary dilation of the image in (a) by a line kernel in the direction of 20°, (c) the result of a regulated dilation of the same image by the same kernel when using a strictness parameter which is greater than one. As can be observed, the regulated dilation by a directional line kernel is capable of filtering lines in a required direction, whereas the ordinary dilation is only capable of thickening the lines in all the other directions.

presents the initial map image, in which there are one side edges (in the lower right direction) of objects in the original image. Fig. 7b demonstrates the result of an ordinary dilation of the image in Fig. 7a by a line kernel in the direction of 20°. Fig. 7c shows the result of a regulated dilation of the same image by the same kernel when using a strictness parameter which is greater than one. As can be observed, the regulated dilation by a directional line kernel is capable of filtering lines in a required direction, whereas the ordinary dilation is only capable of thickening the lines in all the other directions. A complete description of the application of regulated morphological operations for directional decomposition of line drawing images is presented in details in Ref. [11].

An example of separation between a character string and a line that intersects it, is presented in Fig. 8. Fig. 8a presents the initial map image, in which there is a rotated character string that is intersected by a line. Fig. 8b presents the result of line removal from the initial image in Fig. 8a. The line is detected by using an iterative regulated erosion operation that uses a line kernel in the direction of the character string (20°), and a strictness which is greater than one. Fig. 8c shows the result of a regulated erosion of the original image by using a line kernel in a perpendicular direction (110°), and a strictness which is greater than one. Fig. 8d presents the separated character string obtained by the union between the results in Fig. 8b and c. As can be observed, the ability of the regulated erosion to filter lines in a given direction, may be used in order to obtain the required separation.

3. Relations between regulated operations and other operations

In this section the relations between the regulated morphological operations and other operations are studied. In particular the relations to the following operations are examined: ordinary morphological operations, linear filters, order statistic filters, and soft morphological operations.

3.1. Relations to ordinary morphological operations

As demonstrated earlier, the ordinary morphological operations may be obtained from the regulated morphological operations when using a strictness of one. In the following propositions it is discussed how the regulated morphological operations may be obtained by a union or intersection of ordinary morphological operations.

Proposition 24. *The regulated erosion may be obtained by a union of ordinary erosions*:

$$A \overset{s}{\ominus} B = \bigcup_{\{D \subseteq B \mid \#D = \overline{s_B}\}} A \ominus D. \tag{50}$$

Proof. In order to prove the proposition we show that if an element x belongs to the regulated erosion it must belong to the union of the ordinary erosions, and that if it belongs to the union of the ordinary erosions it must belong to the regulated erosion. Assume first that $x \in A \overset{s}{\ominus} B$. Therefore, $\#(A^c \cap (B)_x) < s$, and so $\#(A \cap (B)_x) \geqslant \overline{s_B}$. As follows that, there exists $D \subseteq B$ for which $\#D = \overline{s_B}$ such that $\#(A \cap (D)_x) = \#D$. Therefore, $\#(A^c \cap (D)_x) = 0$ and so by using Eq. (16) we get that $x \in A \ominus D$. Assume now that $x \in \bigcup_{\{D \subseteq B \mid \#D = \overline{s_B}\}} A \ominus D$. Therefore there exists $D \subseteq B$ for which $\#D = \overline{s_B}$ and $x \in A \ominus D$. By using Eq. (16) we get that $\#(A^c \cap (D)_x) = 0$, and so $\#(A^c \cap (B)_x) \leqslant \#B - \#D = s - 1$. Thus $\#(A^c \cap (B)_x) < s$, and so by using Eq. (17) we get that $x \in A \overset{s}{\ominus} B$. □

Proposition 25. *The regulated dilation may be obtained by an intersection of ordinary dilations*:

$$A \overset{s}{\oplus} B = \bigcap_{\{D \subseteq B \mid \#D = \overline{s_B}\}} A \oplus D. \tag{51}$$

Proof. By developing the left side of the proposition based on the duality between the regulated erosion and dilation, and using Eq. (50) we get $A \overset{s}{\oplus} B = (A^c \overset{s}{\ominus} \breve{B})^c = (\bigcup_{\{D \subseteq \breve{B} \mid \#D = \overline{s_B}\}} A^c \ominus D)^c = \bigcap_{\{D \subseteq \breve{B} \mid \#D = \overline{s_B}\}} (A^c \ominus D)^c = \bigcap_{\{D \subseteq B \mid \#D = \overline{s_B}\}} (A^c \ominus \breve{D})^c = \bigcap_{\{D \subseteq B \mid \#D = \overline{s_B}\}} A \oplus D$. □

Proposition 26. *The regulated erosion may be obtained by an intersection of ordinary dilations*:

$$A \overset{s}{\ominus} B = \bigcap_{\{D \subseteq B \mid \#D = s\}} A \oplus \breve{D}. \tag{52}$$

Proof. By developing the left side of the proposition based on Eq. (31) and (51) we get $A \overset{s}{\ominus} B = A \overset{\overline{s_B}}{\oplus} \breve{B} = \bigcap_{\{D \subseteq \breve{B} \mid \#D = s\}} A \oplus D = \bigcap_{\{D \subseteq B \mid \#D = s\}} A \oplus \breve{D}$. □

Proposition 27. *The regulated dilation may be obtained by a union of ordinary erosions*:

$$A \overset{s}{\oplus} B = \bigcup_{\{D \subseteq B \mid \#D = s\}} A \ominus \breve{D}. \tag{53}$$

Proof. By developing the left side of the proposition based on Eq. (30) and (50) we get $A \overset{s}{\oplus} B = A \overset{\overline{s_B}}{\ominus} \breve{B} = \bigcup_{\{D \subseteq \breve{B} \mid \#D = s\}} A \ominus D = \bigcup_{\{D \subseteq B \mid \#D = s\}} A \ominus \breve{D}$. □

3.2 Linear filtering interpretation

Given two images $\underline{A} \equiv \{\underline{A}(k,l)\}_{k,l=-M}^{M}$ and $\underline{B} \equiv \{\underline{B}(k,l)\}_{k,l=-P}^{P}$ where $P < M$, the linear filtering of $\underline{A}$ by $\underline{B}$ is given by the linear convolution between them: $\{\underline{A}(k,l) * \underline{B}(k,l)\}_{k,l=-M}^{M}$. The linear convolution between $\underline{A}(k,l)$ and $\underline{B}(k,l)$ is defined by

$$\underline{A}(k,l) * \underline{B}(k,l) \equiv \sum_{m=-P}^{P} \sum_{n=-P}^{P} \underline{B}(m,n)\underline{A}(k-m,l-n) \tag{54}$$

where it is assumed that $\underline{A}(k,l)$ is zero-padded so that $\underline{A}(k,l) = 0$ for each k, l that is not in the range $[-M, M]$.

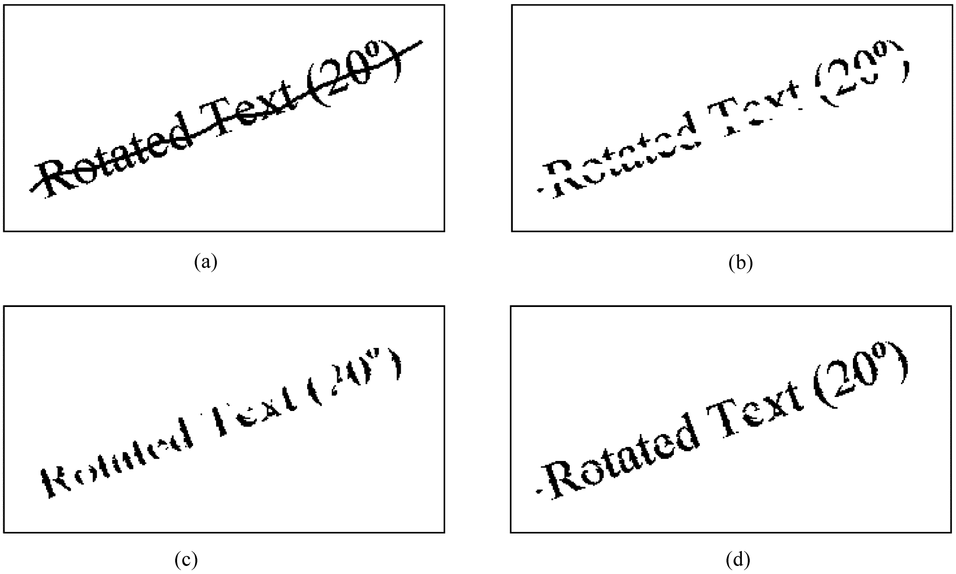


Fig. 8. Demonstration of separation between a character string and an intersecting line by a regulated erosion operation. (a) The initial image of a rotated character string which is intersected by a line, (b) the result of line removal from the initial image in (a). The line is detected by using an iterative regulated erosion operation that uses a line kernel in the direction of the character string (20°), and a strictness which is greater than one. (c) The result of a regulated erosion of the original image by using a line kernel in a perpendicular direction (110°), and a strictness which is greater than one. (d) The separated character string obtained by the union between the results in (b) and (c). As can be observed, the ability of the regulated erosion to filter lines in a given direction, may be used in order to obtain the required separation.

Definition 28. The respective set A of the binary image $\underline{A} \equiv \{\underline{A}(k,l)\}_{k,l=-M}^{M}$ is defined by

$$A \equiv \{(k,l) \mid k,l \in [-M,M],\ \underline{A}(k,l) = 1\} \tag{55}$$

The binary image $\underline{A}$ is called the respective image of the set A.

Lemma 29. *Given two sets $A, B \subset \mathbb{Z}^2$, the cardinality of the intersection between A and the reflected B shifted by (k,l) may be obtained as the value at location (k,l) of the linear convolution between the respective images $\underline{A}$ and $\underline{B}$*:

$$\#(A \cap (\check{B})_{(k,l)}) = \underline{A}(k,l) * \underline{B}(k,l). \tag{56}$$

Proof. By developing the right side of the lemma according to the linear convolution definition we get $\underline{A}(k,l) * \underline{B}(k,l) = \sum_{m=-P}^{P}\sum_{n=-P}^{P}\underline{B}(m,n)\underline{A}(k-m,l-n) = \sum_{m=-P+k}^{P+k}\sum_{n=-P+l}^{P+l}\underline{B}(-m+k,-n+l)\underline{A}(m,n) = \sum_{(-m+k,-n+l)\in B}\underline{A}(m,n) = \sum_{(m,n)\in(\check{B})_{(k,l)}}\underline{A}(m,n) = \#(A \cap (\check{B})_{(k,l)})$. □

Proposition 30. *Given two sets $A, B \subset \mathbb{Z}^2$, the regulated dilation and erosion of A by B may be obtained by thresholding the linear convolution between the respective binary images $\underline{A}$ and $\underline{B}$*:

$$A \overset{s}{\oplus} B = \{(k,l) \mid \underline{A}(k,l) * \underline{B}(k,l) \geqslant s\}, \tag{57}$$

$$A \overset{s}{\ominus} B = \{(k,l) \mid \underline{A}^c(k,l) * \underline{\check{B}}(k,l) < s\}. \tag{58}$$

Proof. Results directly from the definitions of the regulated dilation and erosion, by using Eq. (56). □

Following the last proposition, it is possible to observe that the non-linear nature of morphological operations is derived by a threshold operation, where for the ordinary morphological operations the threshold is 1, and for the regulated morphological operations the threshold may be higher. It should be noted that the properties in this section are discussed for sets in $\mathbb{Z}^2$ in order to simplify the transcription of indexes. These properties can be easily extended to sets in $\mathbb{Z}^N$.

3.3 Relations to order-statistic filters

The mth order statistic of a set of scalars A, is defined by

$$OS^{(m)}(A) \equiv m\text{th smallest of } (\{a \mid a \in A\}). \tag{59}$$

Given a set F, it is possible to define a membership function for it by

$$f(x) \equiv \begin{cases} 1 & \text{if } x \in F, \\ 0 & \text{otherwise}. \end{cases} \quad (60)$$

Based on the membership function $f(x)$, it is possible to represent the regulated dilation and erosion of the set F by using the order statistic operation.

Proposition 31. *The regulated dilation of F by B with a strictness of s may be evaluated by using an order-statistic operation*:

$$F \overset{s}{\oplus} B = \{x \mid OS^{(\overline{s_B})}(\{f(b) \mid b \in (\breve{B})_x\}) = 1\}. \quad (61)$$

Proof. By developing the left side of the proposition based on the regulated dilation definition, we get $F \overset{s}{\oplus} B = \{x \mid \#(F \cap (\breve{B})_x) \geqslant s\} = \{x \mid (\sum_{b \in (\breve{B})_x} f(b)) \geqslant s\} = \{x \mid s\text{th largest of } (\{f(b) \mid b \in (\breve{B})_x\}) = 1\} = \{x \mid \overline{s_B}\text{th smallest of } (\{f(b) \mid b \in (\breve{B})_x\}) = 1\} = \{x \mid OS^{(\overline{s_B})}(\{f(b) \mid b \in (\breve{B})_x\}) = 1\}$. □

Proposition 32. *The regulated erosion of F by B with a strictness of s may be evaluated by using an order-statistic operation*:

$$F \overset{s}{\ominus} B = \{x \mid OS^{(s)}(\{f(b) \mid b \in (B)_x\}) = 1\}. \quad (62)$$

Proof. By developing the left side of the proposition based on Eqs. (31) and (61), and using the fact that the complement of $\overline{s_B}$ with respect to B is s, we get that $F \overset{s}{\ominus} B = F \overset{\overline{s_B}}{\oplus} \breve{B} = \{x \mid OS^{(s)}(\{f(b) \mid b \in (B)_x\}) = 1\}$. □

As described in Ref. [6], the ordinary erosion and dilation may be represented by using minimum and maximum operations, respectively. Propositions 31 and 32 demonstrate that the regulated morphological operations obtain a controlled strictness with respect to the ordinary operations by compromising between the minimum and maximum operations, where such a compromise results in an order-statistic operation.

3.4. Relations to soft morphological operations

The soft morphological operations are performed by using a structuring system $[B, A, r]$ in which A is a hard kernel set, $(B \backslash A)$ is a soft kernel set, and r is an order parameter. The symbol $\backslash$ is used in this context to represent the set difference operation. The soft morphological dilation and erosion of the function f by the structuring system $[B, A, r]$ are defined [3] respectively by:

$$(f \oplus [B, A, r])(x) \equiv r\text{th largest of } (\{r \diamond f(a) \mid a \in A_x\} \cup \{f(b) \mid b \in (B \backslash A)_x\}) \quad (63)$$

$$(f \ominus [B, A, r])(x) \equiv r\text{th smallest of } (\{r \diamond f(a) \mid a \in A_x\} \cup \{f(b) \mid b \in (B \backslash A)_x\}) \quad (64)$$

where in these definitions the symbol $\diamond$ represents a repetition operator defined by

$$r \diamond x \equiv (\overbrace{x, \ldots, x}^{r \text{ times}}).$$

By using the membership function $f(x)$ of the set F as defined in Eq. (60), the soft dilation and erosion of the set F may be obtained respectively by,

$$F \oplus [B, A, r] \equiv \{x \mid (f \oplus [B, A, r])(x) = 1\}, \quad (65)$$

$$F \ominus [B, A, r] \equiv \{x \mid (f \ominus [B, A, r])(x) = 1\}. \quad (66)$$

It should be noted that the definitions of the soft dilation and erosion do not contain a reflection of the kernel. Therefore, when using a non-isotropic kernel, the application of a sequence of soft erosion-dilation or soft dilation-erosion will generate a shift of the shapes in the original image.

The following propositions discuss the relations between the regulated morphological operations, and the soft morphological operations, and it is shown that the soft morphological operations may be obtained by a combination of ordinary morphological operations (that use the hard kernel set) and regulated morphological operations (that use the soft kernel set). Therefore, it is possible to observe that the regulated morphological operations define the nature of softness of the soft morphological operations.

Proposition 33. *The soft dilation of the set F by the structuring system* $[B, A, r]$ *may be obtained by a union between an ordinary dilation* (*that uses the hard kernel set*) *and a regulated dilation* (*that uses the soft kernel set*):

$$F \oplus [B, A, r] = (F \oplus \breve{A}) \cup (F \overset{r}{\oplus} (\breve{B} \backslash \breve{A})) \quad (67)$$

Proof. Assume first that $x \in F \oplus [B, A, r]$. Examining the definition of the soft dilation, it is possible to observe that if there exists $a \in A_x$ such that $f(a) = 1$, the repetition of $f(a)$ r times causes the r th largest value in the definition of the soft dilation to be 1. Therefore, by using Eq. (2), we get that $x \in (F \oplus \breve{A})$ is a sufficient condition to satisfy: $x \in F \oplus [B, A, r]$. When that condition is not satisfied, that is $f(a) = 0$ for all $a \in A_x$, the rth largest value in the definition of the soft dilation is determined by the elements $b \in (\check{B}\backslash\check{A})_x$. It follows that in that case it must be that the number of elements $b \in (\check{B}\backslash\check{A})_x$ for which $f(b) = 1$ is greater than or equal to r, so that the rth largest value in the soft dilation definition will be 1. Therefore, by using Eq. (3), we get that in such a case it must be that $x \in (F \overset{r}{\oplus} (\check{B}\backslash\check{A}))$. When combining this condition with the previous condition we obtain that $x \in ((F \oplus \check{A}) \cup (F \overset{r}{\oplus} (\check{B}\backslash\check{A})))$. In a similar way, when assuming that $x \in ((F \oplus \check{A}) \cup (F \overset{r}{\oplus} (\check{B}\backslash\check{A})))$, it is possible to show that it must be that $x \in F \oplus [B, A, r]$. □

Proposition 34. *The soft erosion of the set F by the structuring system* $[B, A, r]$ *may be obtained by an intersection of an ordinary erosion* (*that uses the hard kernel set*) *and a regulated erosion* (*that uses the soft kernel set*):

$$F \ominus [B, A, r] = (F \ominus A) \cap (F \overset{r}{\ominus} (B\backslash A)). \tag{68}$$

Proof. Assume first that $x \in F \ominus [B, A, r]$. Examining the definition of the soft erosion, it is possible to observe that if there exists $a \in A_x$ such that $f(a) = 0$, the repetition of $f(a)$ r times causes the rth smallest value in the definition of the soft erosion to be 0, with contradiction to the assumption that $x \in F \ominus [B, A, r]$. Therefore, it must be that $f(a) = 1$ for all $a \in A_x$, and so by using Eq. (16), we get that it must be that $x \in (F \ominus A)$. Since for all $a \in A_x$ it must be that $f(a) = 1$, the rth smallest value in the definition of the soft erosion is determined by the elements $b \in (B\backslash A)_x$. Following that, it must be that the number of elements $b \in (B\backslash A)_x$ for which $f(b) = 0$ is less than r, so that the r-th smallest value in the soft erosion definition will be 1. Therefore, by using Eq. (17), we get that it must be true that $x \in (F \overset{r}{\ominus} (B\backslash A))$. When combining this condition with the previous condition we obtain $x \in ((F \ominus A) \cap (F \overset{r}{\ominus} (B\backslash A)))$. In a similar way, when assuming that $x \in ((F \ominus A) \cap (F \overset{r}{\ominus} (B\backslash A)))$, it is possible to show that it must be that $x \in F \ominus [B, A, r]$. □

From Eqs. (67) and (68), we get that $F \oplus [B, A, r] \supseteq (F \oplus \check{A})$ and $F \ominus [B, A, r] \subseteq (F \ominus A)$. Therefore, it is possible to conclude that the soft part of the soft dilation may only increase the results of the ordinary dilation $(F \oplus \check{A})$, and that the soft part of the soft erosion may only decrease the results of the ordinary erosion $(F \ominus A)$.

4. Compound regulated morphological operations

Idempotency of a filter means that a basic property of the signal is filtered. While in general, it is difficult to define non-linear filters which are idempotent, it is possible to define regulated open and close operations that are idempotent by extending the fitting interpretation of the ordinary open and close. In this section regulated open and close operations are defined. The properties of these operations are discussed, and it is shown that the ordinary open and close operations may be obtained as a special case of the regulated operations.

4.1. Regulated close

The regulated close is defined by extending the fitting interpretation of the ordinary close. By extending the fitting interpretation of the ordinary close, some basic characteristics of the shape are filtered, and so the obtained regulated close is idempotent. The strictness parameter of the regulated close controls the strength of the close.

Definition 35. The regulated close of A by B with a strictness of s is defined by

$$A \overset{s}{\bullet} B \equiv ((A \underline{\overset{s}{\oplus}} B) \ominus B) \cup A \tag{69}$$

where $s \in [1, \#B]$.

Since in this definition the result is unified with the original set, the defined operation is necessarily extensive (for an arbitrary kernel and strictness).

Fig. 9 demonstrates the fitting interpretation of the ordinary and regulated close. Figs. 9a and b presents the original shape and the kernel set respectively, where the black square indicates the origin of the kernel. Fig. 9c presents the result of an ordinary close in which the elements that are added to the closed shape are elements that are bounded between all the possible shifts of the reflected kernel that do not intersect the shape. Fig. 9d presents the result of a regulated close with a strictness of two in which the elements that are added to the closed shape are elements that are bounded between all the possible shifts of the reflected kernel for which the size of the intersection with the shape is less than two. The elements that were added by the close are marked in these figures in light gray. As can be observed, the result obtained by the regulated close is smaller than the result obtained by the ordinary close, since in the regulated close the shifted kernels may penetrate deeper into the shape, and so the area bounded between them is smaller.

Proposition 36. *The ordinary close is obtained from the regulated close when* $s = 1$:

$$A \bullet B = A \overset{1}{\bullet} B. \tag{70}$$

Proof. By developing the right side of the proposition according to the definition of the regulated close, and using the facts that ordinary close is extensive, and that $A \underline{\overset{1}{\oplus}} B = A \oplus B$, we get: $A \overset{1}{\bullet} B = ((A \underline{\overset{1}{\oplus}} B) \ominus B) \cup A = ((A \oplus B) \ominus B) \cup A = (A \bullet B) \cup A = (A \bullet B)$. □

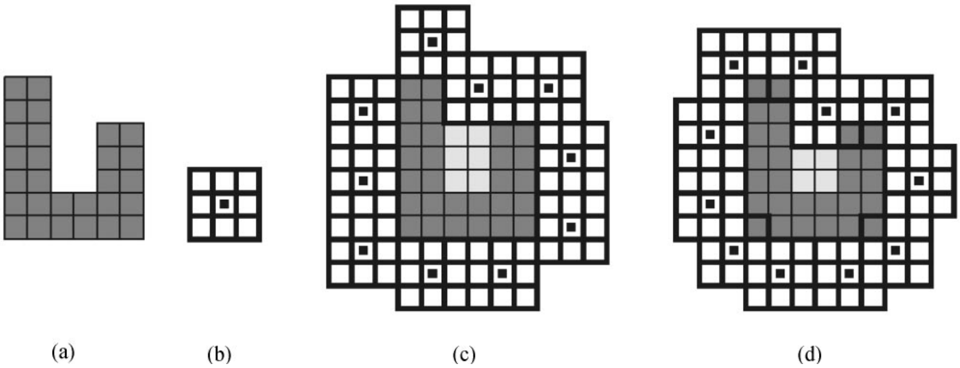


Fig. 9. The fitting interpretation of the ordinary and regulated close. (a) The original shape (b) the kernel set, (c) the result of an ordinary close in which the elements that are added to the closed shape are elements that are bounded between all the possible shifts of the reflected kernel that do not intersect the original shape. (d) The result of a regulated close with a strictness of two in which the elements that are added to the closed shape are elements that are bounded between all the possible shifts of the reflected kernel for which the size of the intersection with the original shape is less than two. The elements that were added by the close operations are marked in light gray. As can be observed, the result obtained by the regulated close is smaller than the result obtained by the ordinary close.

Proposition 37. *The regulated close is decreasing with respect to the strictness s*:

$$A \overset{s1}{\bullet} B \subseteq A \overset{s2}{\bullet} B \;\Leftrightarrow\; s1 \geqslant s2. \tag{71}$$

Proof. Assume first that $s1 \geqslant s2$. Thus, by using Eq. (34), we get that $(A \underline{\overset{s1}{\oplus}} B) \subseteq (A \underline{\overset{s2}{\oplus}} B)$. Therefore $((A \underline{\overset{s1}{\oplus}} B) \ominus B) \cup A \subseteq ((A \underline{\overset{s2}{\oplus}} B) \ominus B) \cup A$, and so $A \overset{s1}{\bullet} B \subseteq A \overset{s2}{\bullet} B$. Assume now that $A \overset{s1}{\bullet} B \subseteq A \overset{s2}{\bullet} B$. We have to show that it must be that $s1 \geqslant s2$. Assume that $s1 < s2$. Thus, by using Eq. (34), we obtain that $(A \underline{\overset{s1}{\oplus}} B) \supseteq (A \underline{\overset{s1}{\oplus}} B)$. Therefore $((A \underline{\overset{s1}{\oplus}} B) \ominus B) \cup A \supseteq ((A \underline{\overset{s2}{\oplus}} B) \ominus B) \cup A$, and so we get $A \overset{s1}{\bullet} B \supseteq A \overset{s2}{\bullet} B$, which contradicts the original assumption that $A \overset{s1}{\bullet} B \subseteq A \overset{s2}{\bullet} B$. Hence $s1 \geqslant s2$. □

Proposition 38. *The regulated close is increasing with respect to the first argument*:

$$A \subseteq B \Rightarrow A \overset{s}{\bullet} K \subseteq B \overset{s}{\bullet} K; \quad s \in [1, \min(\#A, \#K)]. \tag{72}$$

Proof. Since $A \subseteq B$, when using the fact that the extensive regulated dilation and the ordinary erosion are increasing with respect to the first argument, we get that $(A \underline{\overset{s}{\oplus}} K) \ominus K \subseteq (B \underline{\overset{s}{\oplus}} K) \ominus K$. Therefore, $((A \underline{\overset{s}{\oplus}} K) \ominus K) \cup A \subseteq ((B \underline{\overset{s}{\oplus}} K) \ominus K) \cup B$, and so by using the definition of the regulated close we get $A \overset{s}{\bullet} K \subseteq B \overset{s}{\bullet} K$. □

Proposition 39. *The regulated close is translation invariant*:

$$(A)_x \overset{s}{\bullet} B = (A \overset{s}{\bullet} B)_x. \tag{73}$$

Proof. By developing the left side of the proposition based on the definition of the regulated close, and using the fact that the extensive regulated dilation and the ordinary erosion are translation invariant, we get: $(A)_x \overset{s}{\bullet} B = (((A)_x \underline{\overset{s}{\oplus}} B) \ominus B) \cup (A)_x = ((A \underline{\overset{s}{\oplus}} B) \ominus B)_x \cup (A)_x = (((A \underline{\overset{s}{\oplus}} B) \ominus B) \cup A)_x = (A \overset{s}{\bullet} B)_x$. □

4.2. Regulated open

The regulated open is defined by extending the fitting interpretation of the ordinary open. By extending the fitting interpretation of the ordinary open, some basic characteristics of the shape are filtered, and so the obtained regulated open is idempotent. The strictness parameter of the regulated open controls the strength of the open.

Definition 40. The regulated open of A by B with a strictness of s is defined by

$$A \overset{s}{\circ} B \equiv ((A \underline{\overset{s}{\ominus}} B) \oplus B) \cap A \tag{74}$$

where $s \in [1, \#B]$.

Since in this definition the result is intersected with the original set, the defined operation is necessarily anti-extensive (for an arbitrary kernel and strictness).

Fig. 10 demonstrates the fitting interpretation of the ordinary and regulated open. Figs. 10a and b presents the original shape and the kernel set respectively, where the black square indicates the origin of the kernel. Fig. 10c presents the result of an ordinary open in which the elements that are removed in order to create the opened shape are elements that are not covered by any shift of the kernel that is contained completely within the shape. Fig. 10d presents the result of a regulated open with a strictness of two in which the elements that are removed in order to create the opened shape are elements that are

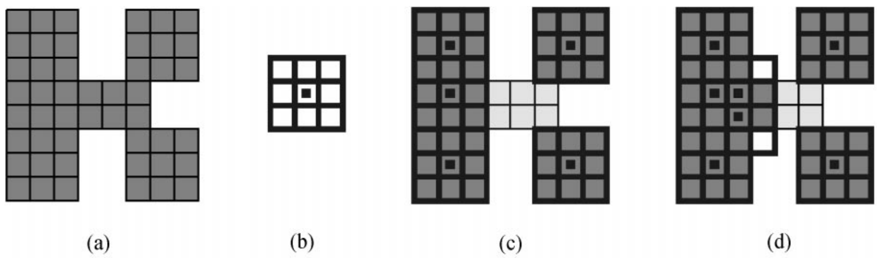


Fig. 10. The fitting interpretation of the ordinary and regulated open. (a) The original shape. (b) The kernel set. (c) The result of an ordinary open in which the elements that are removed from the shape are elements that are not covered by any shift of the kernel that is contained completely within the original shape. (d) The result of a regulated open with a strictness of two in which the elements that are removed from the shape are elements that are not covered by any shift of the kernel that has at most one element common with the background. The elements that were removed by the open operations are marked in light gray. As can be observed, the result obtained by the regulated open is larger than the result obtained by the ordinary open.

not covered by any shift of the kernel that has less than two elements common with the background set. The elements that were removed by the open are marked in these figures in light gray. As can be observed, the result obtained by the regulated open is larger than the result obtained by the ordinary open, since in the regulated open the shifted kernels may have one pixel outside the shape, and so they cover more elements of the original shape.

Proposition 41. *The ordinary open is obtained from the regulated open when* $s = 1$:

$$A \circ B = A \overset{1}{\circ} B. \tag{75}$$

Proof. By developing the right side of the proposition according to the definition of the regulated open, and using the facts that ordinary open is anti-extensive, and that $A \underline{\overset{1}{\ominus}} B = A \ominus B$, we get $A \overset{1}{\circ} B = ((A \underline{\overset{1}{\ominus}} B) \oplus B) \cap A = ((A \ominus B) \oplus B) \cap A = (A \circ B) \cap A = (A \circ B)$. □

Proposition 42. *The regulated open and close are dual in the same sense that exists for the ordinary open and close*:

$$A \overset{s}{\bullet} B = (A^{c} \overset{s}{\circ} \breve{B})^{c}. \tag{76}$$

Proof. By developing the left side of the proposition according to the regulated close definition we get $A \overset{s}{\bullet} B = ((A \underline{\overset{s}{\oplus}} B) \ominus B) \cup A = ((A^{c} \underline{\overset{s}{\ominus}} \breve{B})^{c} \ominus B) \cup A = ((A^{c} \underline{\overset{s}{\ominus}} \breve{B}) \oplus \breve{B})^{c} \cup A = (((A^{c} \underline{\overset{s}{\ominus}} \breve{B}) \oplus \breve{B}) \cap A^{c})^{c} = (A^{c} \overset{s}{\circ} \breve{B})^{c}$. □

Proposition 43. *The regulated open is increasing with respect to the strictness s*:

$$A \overset{s1}{\circ} B \subseteq A \overset{s2}{\circ} B \Leftrightarrow s1 \leqslant s2. \tag{77}$$

Proof. By using Eq. (71) we get that $A^{c} \overset{s2}{\bullet} \breve{B} \subseteq A^{c} \overset{s1}{\bullet} \breve{B} \Leftrightarrow s2 \geqslant s1$. Therefore, $(A^{c} \overset{s2}{\bullet} \breve{B})^{c} \supseteq (A^{c} \overset{s1}{\bullet} \breve{B})^{c} \Leftrightarrow s2 \geqslant s1$, and so by using the duality proposition of the regulated open and close we get that $A \overset{s2}{\circ} B \supseteq A \overset{s1}{\circ} B \Leftrightarrow s2 \geqslant s1$. □

Proposition 44. *The regulated open is increasing with respect to the first argument*:

$$A \subseteq B \Rightarrow A \overset{s}{\circ} K \subseteq B \overset{s}{\circ} K; \quad s \in [1, \# K]. \tag{78}$$

Proof. Since $A \subseteq B$ it follows that $A^{c} \supseteq B^{c}$. When using the fact that the regulated close is increasing with respect to the first argument, we get that $A^{c} \overset{s}{\bullet} \breve{K} \supseteq B^{c} \overset{s}{\bullet} \breve{K}$. Therefore, $(A^{c} \overset{s}{\bullet} \breve{K})^{c} \subseteq (B^{c} \overset{s}{\bullet} \breve{K})^{c}$, and so by using the duality proposition between the regulated open and close, we get $A \overset{s}{\circ} K \subseteq B \overset{s}{\circ} K$. □

Proposition 45. *The regulated open is translation invariant*:

$$(A)_{x} \overset{s}{\circ} B = (A \overset{s}{\circ} B)_{x}. \tag{79}$$

Proof. By developing the left side of the proposition based on the definition of the regulated open, and using the fact that the anti-extensive regulated erosion and the ordinary dilation are translation invariant, we get $(A)_{x} \overset{s}{\circ} B = (((A)_{x} \underline{\overset{s}{\ominus}} B) \oplus B) \cap (A)_{x} = ((A \underline{\overset{s}{\ominus}} B) \oplus B)_{x} \cap (A)_{x} = (((A \underline{\overset{s}{\ominus}} B) \oplus B) \cap A)_{x} = (A \overset{s}{\circ} B)_{x}$. □

Fig. 11 presents an example of regulated open and close operations in which the original shape and the kernel set are identical to those used in Fig. 3. Figs 11a–e present the results of a regulated open with a strictness of 1–5, respectively, and Figs. 11f–j presents the results of a regulated close with a strictness of 1–5, respectively. The elements that were removed from the original shape are marked in these figures in light gray.

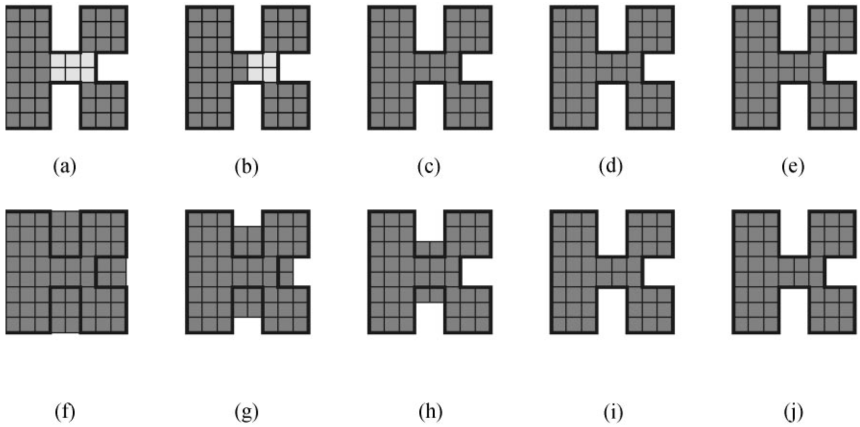


Fig. 11. Demonstration of the regulated open and close operations. (a)–(e) The results of a regulated open with a strictness of 1–5, respectively. (f)–(j) The results of a regulated close with a strictness of 1–5, respectively. The original shape and the kernel set that are used in this example are identical to those used in Fig. 3. The elements that were removed from the original shape are marked in light gray.

4.3. The fitting interpretation of the regulated open and close

In the beginning of Section 4 it was stated that the regulated open and close operations extend the fitting interpretation of the ordinary open and close operations. The following propositions formulate the fitting interpretation of the regulated open and close operations.

Proposition 46. *The regulated open of a shape A by a kernel B with a strictness of s may be interpreted as the intersection between the shape and the union of all the possible shifts of the kernel for which the intersection with the shape is big enough and the origin of the kernel is included in the shape*:

$$A \overset{s}{\circ} B = \left(\bigcup_{\{x\in A \mid \#(A^c \cap (B)_x) < s\}} (B)_x \right) \cap A\,. \tag{80}$$

Proof. According to the definition of the anti-extensive regulated erosion: $A \overset{s}{\underline{\ominus}} B = (A \overset{s}{\ominus} B) \cap A = \{x \in A \mid \#(A^c \cap (B)_x) < s\}$. When assigning this into the definition of the regulated open, and using the union interpretation of the ordinary dilation we get: $A \overset{s}{\circ} B = ((A \overset{s}{\underline{\ominus}} B) \oplus B) \cap A = (\{x \in A \mid \#(A^c \cap (B)_x) < s\} \oplus B) \cap A = (\bigcup_{\{x\in A \mid \#(A^c\cap(B)_x)<s\}} (B)_x) \cap A$. □

Based on the last proposition, the fitting interpretation of the regulated open may be visualized as moving the kernel inside the shape as close as possible to its borders, where the kernel may get out of the shape up to some extent, and then eliminating border elements that were not covered by at least one shift of the kernel. An example of the fitting interpretation of the regulated open was already presented in Fig. 10.

Proposition 47. *The regulated close of a shape A by a kernel B with a strictness of s may be interpreted as the union of the shape and the intersection between all the possible shifts of the reflected and complemented kernel for which the intersection of the reflected kernel with the shape is small enough and the origin of the kernel is not included in the shape*:

$$A \overset{s}{\bullet} B = \left(\bigcap_{\{x\in A^c \mid \#(A\bigcap(\breve{B})_x) < s\}} ((\breve{B})_x)^c \right) \cup A\,. \tag{81}$$

Proof. By using Eq. (80) and the duality between the regulated open and close, we get $A \overset{s}{\bullet} B = (A^c \overset{s}{\circ} \breve{B})^c = ((\bigcup_{\{x\in A^c \mid \#(A\bigcap(\breve{B})_x)<s\}} (\breve{B})_x) \cap A^c)^c = (\bigcup_{\{x\in A^c \mid \#(A\bigcap(\breve{B})_x)<s\}} (\breve{B})_x)^c \cup A = (\bigcap_{\{x\in A^c \mid \#(A\bigcap(\breve{B})_x)<s\}} ((\breve{B})_x)^c) \cup A$. □

Based on the last proposition, the fitting interpretation of the regulated close may be visualized as moving the reflected kernel outside the shape as close as possible to its borders, where the reflected kernel may get into the shape up to some extent, and then adding border elements that were not covered by at least one shift of the reflected kernel. An example of the fitting interpretation of the regulated close was already presented in Fig. 9.

4.4. Idempotency of the regulated open and close

In the beginning of Section 4 it was stated that the regulated open and close operations are idempotent. The following propositions provide a proof of that statement.

Proposition 48. *The extensive regulated dilation of a set is not influenced by a regulated close operation that is performed on the set before the extensive regulated dilation* (*with the same kernel and strictness parameter*):

$$(A \overset{s}{\bullet} B) \underline{\overset{s}{\oplus}} B = A \underline{\overset{s}{\oplus}} B. \tag{82}$$

Proof. Since the regulated close is extensive, it is possible to construct a set D that contains all the elements that are added to a shape during the regulated close operation: $D \equiv (A \overset{s}{\bullet} B) \cap A^c$. By using the set D the regulated close of the shape is obtained by the union: $A \cup D$. Assume first that $D = \emptyset$, in that case $A \overset{s}{\bullet} B = A$, and so $(A \overset{s}{\bullet} B) \underline{\overset{s}{\oplus}} B = A \underline{\overset{s}{\oplus}} B$. Assume now that $D \neq \emptyset$, in that case there exists an element $x \in D$. According to the definition of D, when $x \in D$ it follows that $x \in A \overset{s}{\bullet} B$ and $x \in A^c$. Since D is not empty, in order to prove the proposition it is sufficient to show that the element x cannot influence the extensive regulated dilation of A (with the same strictness s). Consequently, it is required to show that: (a) The element x belongs to the extensive regulated dilation of A; (b) the presence of the element x does not cause the addition of new elements to the extensive regulated dilation of A.

Part (a) is proved by using the regulated close definition. Since $x \in A \overset{s}{\bullet} B$, according to the regulated close definition it follows that $x \in ((A \underline{\overset{s}{\oplus}} B) \ominus B) \cup A$. When combining that with the fact that $x \in A^c$ we get that $x \in (A \underline{\overset{s}{\oplus}} B) \ominus B$. Therefore, since the ordinary erosion is anti-extensive, it follows that $x \in A \underline{\overset{s}{\oplus}} B$. Part (b) is proved by using the fitting interpretation of the regulated close. Since $x \in A \overset{s}{\bullet} B$, according to Eq. (81) it follows that $x \in (\bigcap_{\{y \in A^c | \#(A \bigcap (\breve{B})_y) < s\}} ((\breve{B})_y)^c) \cup A$. When combining that with the fact that $x \in A^c$ we get that $x \in \bigcap_{\{y \in A^c | \#(A \bigcap (\breve{B})_y) < s\}} ((\breve{B})_y)^c$. Therefore $x \notin \bigcup_{\{y \in A^c | \#(A \bigcap (\breve{B})_y) < s\}} (\breve{B})_y$, and so for all the possible shifts $y \in A^c$ for which $x \in (\breve{B})_y$ it must be that $\#(A \cap (\breve{B})_y) \geqslant s$. Since any shift $y \in A^c$ of the reflected kernel that includes x has an intersection with A with at least s elements, it will be included in the extensive regulated dilation of A whether the element x is present or not. Thus, the presence of the element x does not cause the addition of new elements to the extensive regulated dilation of A. □

Propositon 49. *The anti-extensive regulated erosion of a set is not influenced by a regulated open operation that is performed on the set before the anti-extensive regulated erosion* (*with the same kernel and strictness parameter*):

$$(A \overset{s}{\circ} B) \underline{\overset{s}{\ominus}} B = A \underline{\overset{s}{\ominus}} B. \tag{83}$$

Proof. By using the duality between the regulated open and close, and Eq. (82), we get $(A \overset{s}{\circ} B) \underline{\overset{s}{\ominus}} B = (A^c \overset{s}{\bullet} \breve{B})^c \underline{\overset{s}{\ominus}} B = ((A^c \overset{s}{\bullet} \breve{B}) \underline{\overset{s}{\oplus}} \breve{B})^c = (A^c \underline{\overset{s}{\oplus}} \breve{B})^c = A \underline{\overset{s}{\ominus}} B$. □

Proposition 50. *The regulated close is idempotent*:

$$(A \overset{s}{\bullet} B) \overset{s}{\bullet} B = A \overset{s}{\bullet} B. \tag{84}$$

Proof. By eroding both sides of Eq. (82) with B and uniting them with A we get that $(((A \overset{s}{\bullet} B) \underline{\overset{s}{\oplus}} B) \ominus B) \cup A = ((A \underline{\overset{s}{\oplus}} B) \ominus B) \cup A$, and so by using the definition of the regulated close we obtain that $(((A \overset{s}{\bullet} B) \underline{\overset{s}{\oplus}} B) \ominus B) \cup A = A \overset{s}{\bullet} B$. By uniting both sides with $A \overset{s}{\bullet} B$ and using the fact that $A \cup (A \overset{s}{\bullet} B) = A \overset{s}{\bullet} B$ (since the regulated close is extensive) we get that $(((A \overset{s}{\bullet} B) \underline{\overset{s}{\oplus}} B) \ominus B) \cup A \overset{s}{\bullet} B = A \overset{s}{\bullet} B$, which according to the definition of the regulated close is equivalent to $(A \overset{s}{\bullet} B) \overset{s}{\bullet} B = A \overset{s}{\bullet} B$. □

Propositon 51. *The regulated open is idempotent*:

$$(A \overset{s}{\circ} B) \overset{s}{\circ} B = A \overset{s}{\circ} B. \tag{85}$$

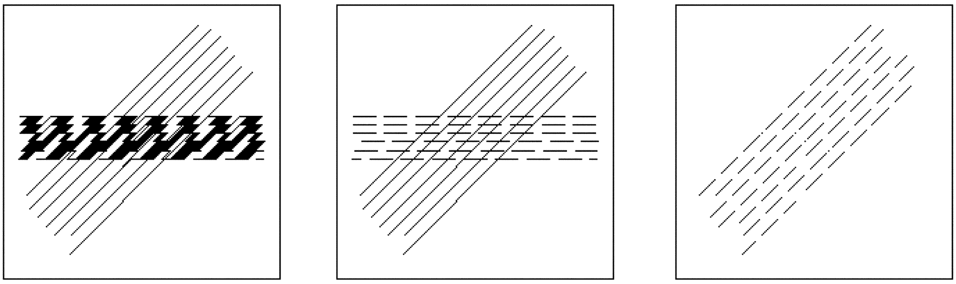
Proof. By using the duality between the regulated open and close, and the fact that the regulated close is idempotent, we get $(A \overset{s}{\circ} B) \overset{s}{\circ} B = (A^c \overset{s}{\bullet} B)^c \overset{s}{\circ} B = ((A^c \overset{s}{\bullet} B) \overset{s}{\bullet} B)^c = (A^c \overset{s}{\bullet} B)^c = A \overset{s}{\circ} B$.

4.5. Examples of the regulated open and close

This section presents some examples of applications of the regulated open and close operations, and demonstrates their ability to improve the results obtained by the ordinary operations when using strictness parameter greater than one. Fig. 12 demonstrates the results of a regulated close and open of the simple dashed lines image, presented in Fig. 5a. Fig. 12a shows the result of an ordinary close of the image by a line kernel in the direction of 45°. Fig. 12b and c presents the result of a regulated close and open, respectively, of the same image by the same line kernel when using a strictness parameter which is greater than one. As can be observed, the regulated close fills the gaps between the diagonal dashed lines without affecting the horizontal dashed lines, whereas the ordinary close generates noise due to the closing of these lines. The regulated open only removes the horizontal dashed lines.

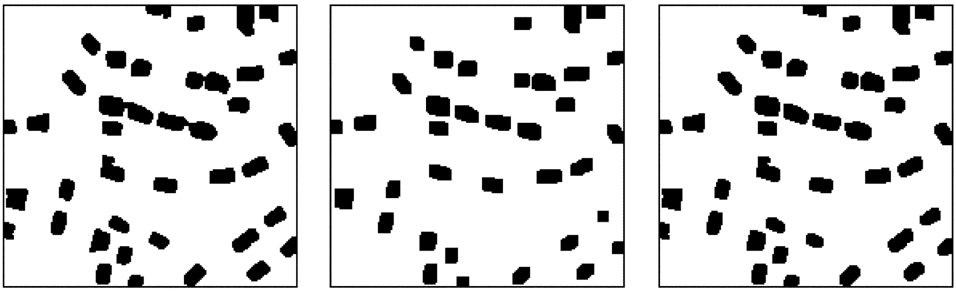
An example of objects separation by a regulated open operation is presented in Fig. 13. Fig. 13a presents the initial map image, containing extracted building marks, in which some of the building marks are touching each other. Fig. 26b demonstrates the result of an ordinary open of the image in Fig. 26a by a 7×7 kernel. Fig. 26c presents the result of a regulated open of the same image by the same kernel when using a strictness parameter of 2. As can be observed, the regulated open operation manages to separate between touching buildings without causing significant changes to their shapes, whereas the ordinary open operation causes the removal of some buildings and changes the shapes of others.

An example of cluttered overlay removal by a regulated open operation is presented in Fig. 14. Fig. 14a presents the initial map image, which contains a cluttered overlay. Fig. 14b demonstrates the result of an ordinary



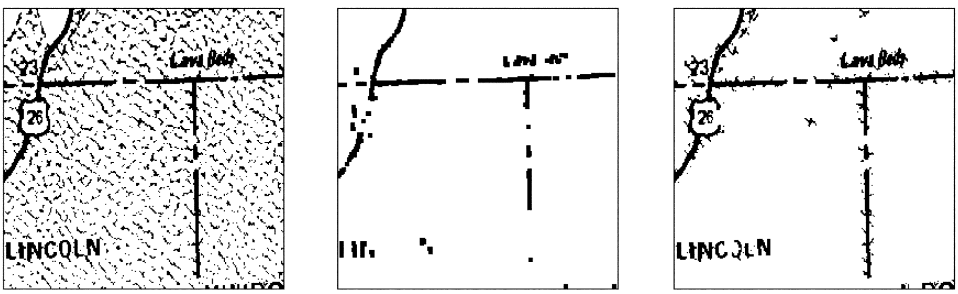
(a) (b) (c)

Fig. 12. Demonstration of the regulated close and open of dashed lines. (a) The result of an ordinary close of the image in Fig. 5a by a line kernel in the direction of 45°. (b)–(c) The result of a regulated close and open, respectively, of the same image by the same kernel when using a strictness parameter which is greater than one. As can be observed, the regulated close fills the gaps between the diagonal dashed lines without affecting the horizontal dashed lines, whereas the ordinary close generates noise due to the close of these lines. The regulated open only removes the horizontal dashed lines.



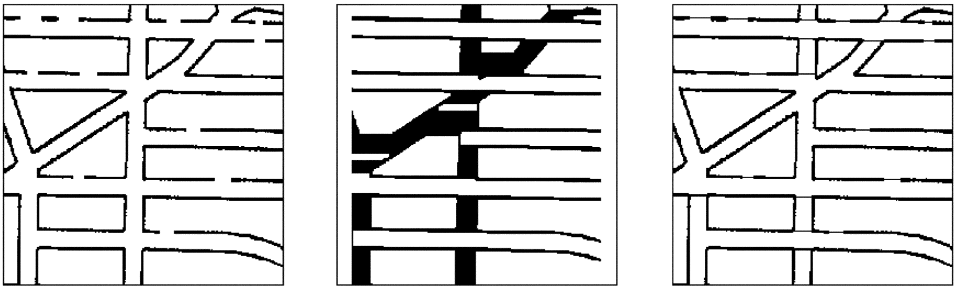
(a) (b) (c)

Fig. 13. Demonstration of objects separation by a regulated open operation. (a) The initial map image, containing extracted building marks, in which some of the building marks are touching each other, (b) the result of an ordinary open of the image in (a) by a 7×7 kernel, (c) the result of a regulated open of the same image by the same kernel when using a strictness parameter of 2. As can be observed, the regulated open operation manages to separate between touching buildings without causing significant changes to their shapes, whereas the ordinary open operation causes the removal of some buildings and changes the shapes of others.



(a) (b) (c)

Fig. 14. Demonstration of a cluttered overlay removal by a regulated open operation. (a) The initial map image, containing a cluttered overlay. (b) The result of an ordinary open of the image in (a) by a 4×4 kernel. The size 4 is the minimal size of a square kernel that is capable of removing the clutter by an ordinary open operation. (c) The result of a regulated open of the same image by a 7×7 kernel when using a strictness parameter of 25. As can be observed, the regulated open operation manages to remove most of the cluttered overlay while keeping the other objects in the image, whereas the ordinary open removes parts of objects with the clutter.



(a) (b) (c)

Fig. 15. Demonstration of lines reconstruction by a regulated close operation. (a) The initial map image, containing extracted lines of roads, in which some parts of lines were erroneously removed due to the separation of touching character strings. (b) The result of an ordinary close of the image in (a) by a horizontal bar kernel in the size of 3×30. (c) The result of a regulated close of the same image by the same kernel when using a strictness parameter of 20. As can be observed, the regulated close operation manages to fill the gaps in the lines without causing excessive filling, whereas the ordinary close operation causes the complete filling of gaps between any pair of parallel vertical lines in the image.

open of the image in Fig. 14a by a 4×4 kernel. The size 4 is the minimal size of a square kernel that is capable of removing the clutter by an ordinary open operation. Fig. 14c shows the result of a regulated open of the same image by a 7×7 kernel when using a strictness parameter of 25. As can be observed, the regulated open operation manages to remove most of the cluttered overlay while keeping the other objects in the image, whereas the ordinary open removes parts of objects with the clutter.

An example of lines reconstruction by a regulated close operation is presented in Fig. 15. Fig. 15a presents the initial map image, containing extracted lines of roads, in which some parts of lines were erroneously removed due to the separation of touching character strings. Fig. 5b demonstrates the result of an ordinary close of the image in Fig. 15a by a horizontal bar kernel in the size of 3×30. Fig. 15c presents the result of a regulated close of the same image by the same kernel when using a strictness parameter of 20. As can be observed, the regulated close operation manages to fill the gaps in the lines without causing excessive filling, whereas the ordinary close operation causes the complete filling of gaps between any pair of parallel vertical lines in the image.

5. Examples of algorithms using regulated morphological operations

By using the regulated operations in existing morphological algorithms with a strictness parameter which is greater than one, it is possible to increase their ability to cope with noise and small intrusions or protrusions on the boundary of shapes. Thus, given an existing morphological algorithm, it is possible to try and improve the results obtained by it by using the regulated morphological operations instead of the ordinary morphological operations with strictness that may be optimized according to some optimization criteria. This section presents some examples of regulating other morphological operations which is based on the basic regulated morphological operations. More examples to applications of regulated morphological operations may be found in [7, 10–12].

5.1. Morphological processing of noisy images

As was stated earlier, the regulated morphological operations are less sensitive to noise with respect to the ordinary morphological operations. Fig. 16 presents an example of morphological processing of a noisy image by using the ordinary morphological operations. Fig. 16a presents the original image, and Fig. 16b presents the noisy image in which 10% of the pixels of the original image were replaced by their complement. The result of an ordinary erosion, dilation, open, and close operations applied to the noisy image are presented in Fig. 16c–f, respectively. The kernel used in these operations is a 5×5 square with the origin at its center. As can be observed these operations are very sensitive to the added noise, and do not obtain their designated task.

Fig. 17 presents an example of morphological processing of the noisy image in Fig. 16b by using the regulated morphological operations. The results of the regulated erosion, regulated dilation, regulated erosion–dilation, and regulated dilation–erosion operations applied to the noisy image with a strictness of 7 are presented in Fig. 17a–d, respectively. The results of the anti-extensive regulated erosion, extensive regulated dilation, regulated open, and regulated close operations applied to the noisy image with a strictness of 7 are presented in Fig. 17e–h,

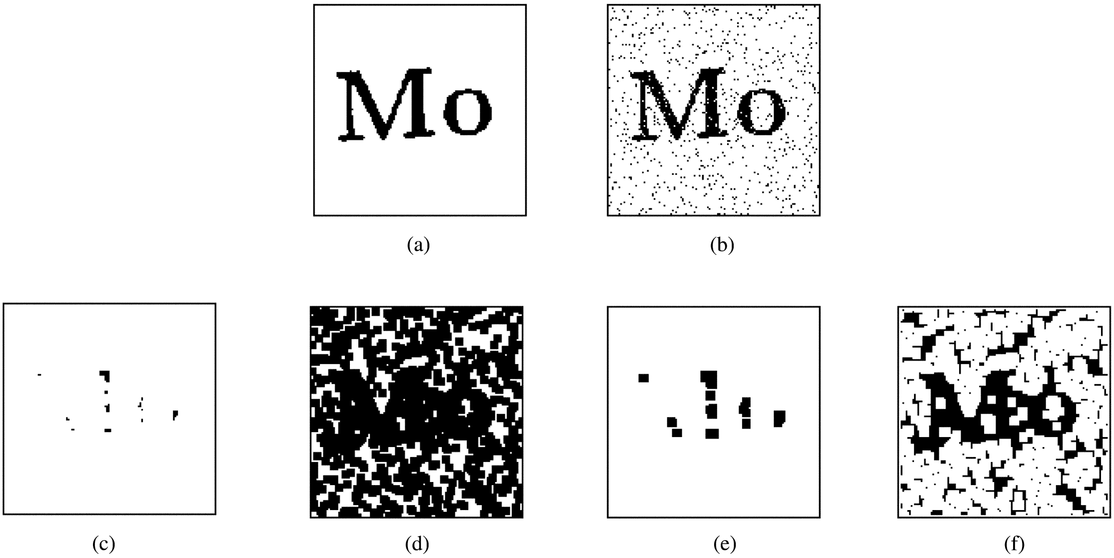


Fig. 16. Demonstration of ordinary morphological operations on a noisy image. (a) The original binary image. (b) The noisy image obtained by replacing 10% of the pixels in the original image with their complement. (c)–(f) The results of ordinary erosion, dilation, open, and close of the noisy image. The kernel used in these operations is a 5×5 square with the origin at its center. As can be observed these operations are very sensitive to the added noise, and do not obtain their designated task.

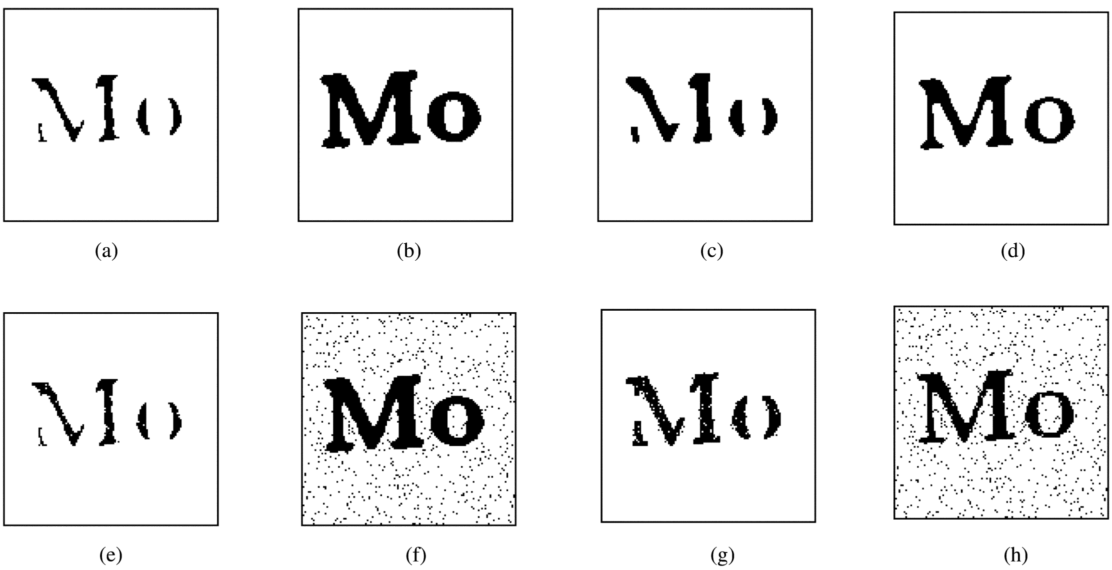


Fig. 17. Demonstration of regulated morphological operations on a noisy image. (a)–(d) The results of regulated erosion, regulated dilation, regulated erosion–dilation, and regulated dilation–erosion, of the noisy image with a strictness of 7. (e)–(h) The results of anti-extensive regulated erosion, extensive regulated dilation, regulated open, and regulated close of the noisy image with a strictness of 7. The original image is presented in Fig. 16b. The kernel used in these operations is a 5×5 square with the origin at its center. As can be observed the regulated erosion and dilation operations perform their designated task while suppressing the added noise, whereas the extensive regulated dilation and the anti-extensive regulated erosion perform their designated task without suppressing the added noise. A similar observation exists for the regulated erosion-dilation, regulated dilation-erosion and the regulated open and close operations.

respectively. The kernel used in these operations is a 5×5 square with the origin at its center. As can be observed, the regulated erosion and dilation operations perform their designated task while suppressing the added noise, whereas the extensive regulated dilation and the anti-extensive regulated erosion perform their designated task without suppressing the added noise. A similar observation exists for the regulated erosion–dilation, regulated dilation–erosion and the regulated open and close operations.

5.2. Regulated hit-or-miss transform

The ordinary hit-or-miss transform [13] finds shifts of the kernel for which the kernel is contained completely within the shape, and its background set does not intersect the shape. A shift for which the kernel and the background sets fit in the shape and its background, respectively, is called a hit. A shift which is not a hit is called a miss. The ability of the regulated morphological operations to cope with noise and small intrusions or protrusions on the boundary of shapes may be used in order to define a regulated hit-or-miss transform which can enable intermediate levels between the two extreme situations of hit or miss.

Definition 52. The regulated hit-or-miss transform of A by (B_1, B_2) with a strictness of $(s1, s2)$ is defined by

$$A \overset{(s1,s2)}{\circledast} (B_1, B_2) \equiv (A \overset{s1}{\underline{\ominus}} B_1) \cap (A^c \overset{s2}{\ominus} B_2) \quad (86)$$

where B_1 is the kernel set, B_2 is the background set, and $s1$ and $s2$ are the strictness parameters used for the kernel and background sets, respectively.

It should be noted that the second term in the definition uses the regulated erosion rather than the anti-extensive regulated erosion since the usage of the anti-extensive regulated erosion would force intersection with A^c in addition to an intersection with A thus resulting in an empty set.

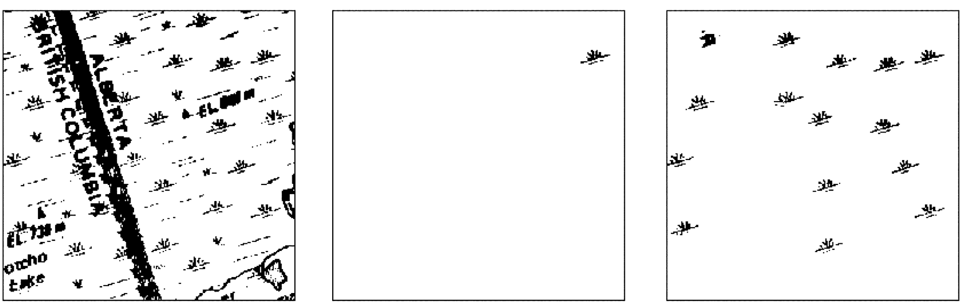
There is a need for two strictness parameters in the regulated hit-or-miss transform since in general the cardinality of the kernel and the background sets may be different. When the origin is included in B_1, the ordinary hit-or-miss transform is obtained from the regulated hit-or-miss transform by using a strictness of (1, 1). When using strictness parameters which are greater than one, imperfect hits are enabled. The regulated hit-or-miss transform may be also evaluated by

$$A \overset{(s1,s2)}{\circledast} (B_1, B_2) \equiv (A \overset{s1}{\underline{\ominus}} B_1) - (A \overset{s2}{\oplus} \check{B}_2) \quad (87)$$

where the operator $-$ represents set difference. Fig. 18 presents an example of using the regulated hit-or-miss transform for a simple template matching. Fig. 18a presents the original image, Fig. 18b presents the result of an ordinary hit-or-miss transform, and Fig. 18c presents the result of a regulated hit-or-miss transform with a strictness of (50, 50). The kernel that was used for these transforms is the shape in Fig. 18b. The background set was obtained by dilating the kernel set with a 5×5 square (with the origin at its center) and then removing elements that belong to the kernel set from the resulting set. In order to obtain a better presentation of the results in Fig. 18b and c, the results of the hit-or-miss transforms were dilated by the kernel set and a 3×3 square (with the origin at its center), and then intersected with the original image. As can be observed the regulated hit-or-miss transform detects more objects than the ordinary hit-or-miss transform which detects only the one object that is identical to the kernel set.

5.3. Regulated skeletonizing operation

The ordinary skeletonizing operation [13] is based on an iterative operation where in each iteration the



(a) (b) (c)

Fig. 18. Demonstration of a regulated hit-or-miss transform. (a) The original image, (b) the result of an ordinary hit-or-miss transform, (c) The result of a regulated hit-or-miss transform with a strictness of (50, 50). As can be observed the regulated hit-or-miss transform detects more objects than the ordinary hit-or-miss transform.

elements that are added to the skeleton of a shape are elements that are not covered by any shift of the kernel that is contained completely within the shape. At the end of each iteration the shape is eroded and the process continues until an empty set is obtained.

Definition 53. The regulated skeletonizing operation of A by B with a strictness of s is defined by

$$S_s(A) \equiv \bigcup_{k=0}^{K} (A \overset{s}{\underline{\ominus}} kB) - ((A \overset{s}{\underline{\ominus}} kB) \circ B) \tag{88}$$

where $A \overset{s}{\underline{\ominus}} kB$ represents k successive erosions of A by B with a strictness of s, the operator $-$ represents set difference, and $K = \max\{k | (A \overset{s}{\underline{\ominus}} kB) \neq \emptyset\}$.

Since when using high strictness values there could be a situation in which the shape is not eroded, an additional condition is introduced for stopping the iterations. The additional condition requires that a change must occur in the shape during each iteration.

The ordinary skeletonizing operation is obtained from the regulated skeletonizing operation when using a strictness of 1. By using a strictness parameter which is greater than one the number of shape elements that are removed at each iteration is reduced, and so a finer progress of the process is obtained. An alternative way to regulate the ordinary skeletonizing operation may be obtained by exchanging the ordinary open operation in Eq. (88) with a regulated open operation. In such a case, when using a strictness parameter which is greater than one, it is possible to cover more elements in the shape by the kernel. Therefore, fewer skeleton elements are found during each iteration, and the resulting skeleton is less connected.

Fig. 19 presents the results of a regulated skeletonizing operation. Fig. 19a presents the original image, Fig. 19b presents the result of an ordinary skeletonizing operation, and Figs. 19c and d presents the results of a regulated skeletonizing operation with strictness parameters of two and three respectively. The kernel used in these operations is a 3×3 square with the origin at its center. As can be observed the regulated skeletonizing operations result in a more connected skeleton.

5.4. Regulated thinning and thickening operations

The ordinary thinning operation [13] of a set is based on a hit-or-miss transform that is used in order to remove elements from the set. The thinning of a shape is obtained by an iterative process in which during each iteration a sequence of thinning operations is performed by using a sequence of kernels that are designed to remove boundary elements from the shape. In order to get a symmetrical thinning of the shape, the sequence of kernels that is used during each iteration is normally composed by a sequence of rotations of a directional kernel. The iterative process continues until no further change occurs in the shape.

Definition 54. The regulated thinning operation of A by (B_1, B_2) with a strictness of $(s1, s2)$ is defined by

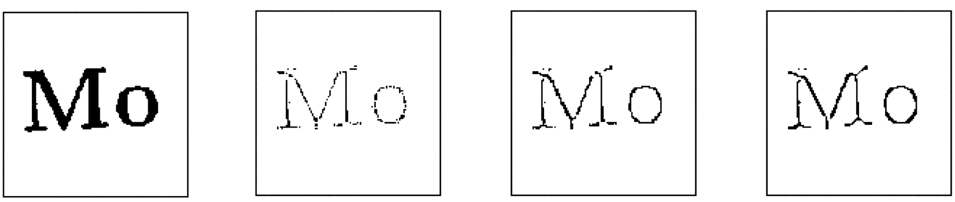
$$A \overset{(s1,s2)}{\otimes} (B_1, B_2) \equiv A - (A \overset{(s1,s2)}{\circledast} (B_1, B_2)) \tag{89}$$

where the operator $-$ represents set difference, B_1 is the kernel set, B_2 is the background set, and $s1$ and $s2$ are the strictness parameters used for the kernel and background sets respectively.

Definition 55. The regulated thinning operation of A by a set of N kernels $\{(B_1^n, B_2^n)\}$ with a strictness of $(s1, s2)$ is defined by

$$A \overset{(s1,s2)}{\otimes} \{(B_1^n, B_2^n)\} \equiv ((\cdots((A \overset{(s1,s2)}{\otimes} (B_1^1, B_2^1)) \overset{(s1,s2)}{\otimes} (B_1^2, B_2^2)) \cdots \overset{(s1,s2)}{\otimes} (B_1^N, B_2^N)). \tag{90}$$

A regulated thinning of a shape is obtained by applying the regulated thinning operation that uses a set of



(a) (b) (c) (d)

Fig. 19. Demonstration of a regulated skeletonizing operation. (a) The original image, (b) The result of an ordinary skeletonizing operation, (c)–(d) The results of a regulated skeletonizing operation with a strictness of 2–3, respectively. As can be observed the regulated skeletonizing operations result in a more connected skeleton.

kernels iteratively, where the kernels are designed to remove edge elements of the shape. The ordinary thinning operation is obtained from the regulated thinning operation when using a strictness of 1. By using a strictness parameter which is greater than one the hit-or-miss operation that is used by the thinning operation becomes less strict, and so the obtained results become less influenced by small intrusions or protrusions on the boundary of the shape.

Fig. 20 presents the results of a regulated thinning operation. Fig. 20a presents the original image, Fig. 20b presents the result of an ordinary thinning operation, and Fig. 20c–d presents the results of a regulated thinning operation with strictness parameters of (2, 1) and (3, 1), respectively. A set of four kernels (B_1^n, B_2^n) was used in this example, where (B_1^1, B_2^1) is given by: $(B_1^1, B_2^1) = (\{(0,0),(1,-1),(1,0),(1,1)\},\{(-1,-1),(-1,0),(-1,1)\})$, and (B_1^2, B_2^2) – (B_1^4, B_2^4) are obtained as rotations of (B_1^1, B_2^1) in 90°, 180°, and 270°, respectively. As can be observed the result of the regulated thinning operation with a strictness of (2, 1) is cleaner with respect to the ordinary thinning operation, whereas the result of the regulated thinning operation with a strictness of (3, 1) extracts only key points in the shape.

The ordinary thickening operation [13] of a set may be obtained by thinning its complement, and taking the complement of the obtained result. The regulated thickening operation may be obtained in a similar way.

Definition 56. The regulated thickening operation of A by (B_1, B_2) with a strictness of $(s1, s2)$ is defined by

$$A \overset{(s1,s2)}{\odot} (B_1, B_2) \equiv (A^c \overset{(s1,s2)}{\otimes} (B_1, B_2))^c \tag{91}$$

where B_1 is the kernel set, B_2 is the background set, and $s1$ and $s2$ are the strictness parameters used for the kernel and background sets, respectively.

The regulated thickening operation of A by a set of N kernels $\{(B_1^n, B_2^n)\}$ with a strictness of $(s1, s2)$ is defined similar to Eq. (90).

Fig. 21 presents the results of a regulated thickening operation. Fig. 21a presents the original image, Fig. 21b presents the result of an ordinary thickening operation, and Fig. 21c and d presents the results of a regulated thickening operation with strictness parameters of (3, 1) and (3, 2), respectively. The set of kernels that is used to obtain the results in this example is identical to the set which was described for the demonstration of the regulated thinning operation in Fig. 20. As can be observed, it is possible to obtain smother results by using the

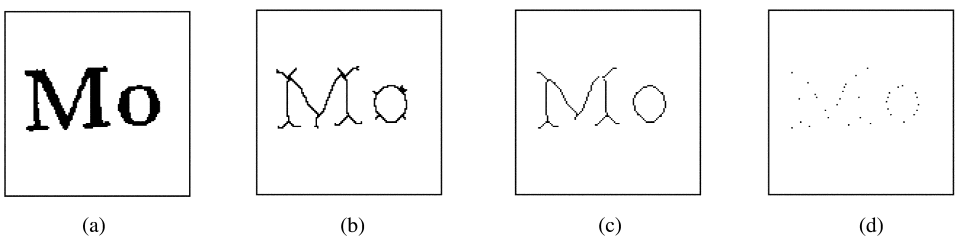


Fig. 20. Demonstration of a regulated thinning operation. (a) The original image. (b) The result of an ordinary thinning operation. (c)–(d) The results of a regulated thinning operation with a strictness of (2, 1) and (3, 1) respectively. As can be observed the result of the regulated thinning operation with a strictness of (2, 1) is cleaner with respect to the ordinary thinning operation, whereas the result of the regulated thinning operation with a strictness of (3, 1) extracts only key points in the shape.

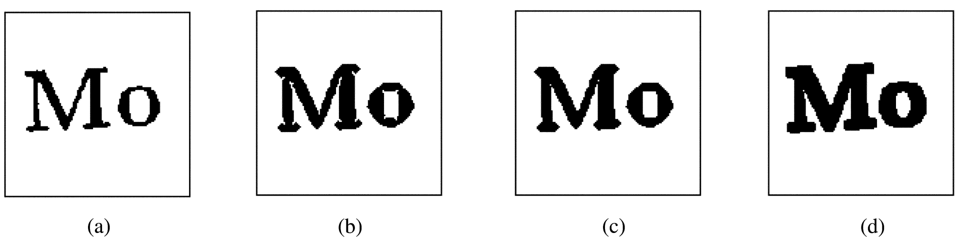


Fig. 21. Demonstration of a regulated thickening operation. (a) The original image, (b) the result of an ordinary thickening operation, (c)–(d) the results of a regulated thickening operation with a strictness of (3, 1) and (3, 2), respectively. As can be observed, it is possible to obtain smother results by using the regulated thickening operation with strictness parameters which are larger than (1, 1).

regulated thickening operation with strictness parameters which are larger than (1, 1).

6. Summary

This paper describes the problem of sensitivity of the ordinary morphological operations to noise and small intrusions or protrusions on the boundary of shapes. This problem is derived by the strict approach taken by the ordinary morphological operations, and so in order to solve the problem, regulated erosion and dilation are defined by extending the fitting interpretation of the ordinary operations. The regulated erosion and dilation have a controllable strictness parameter, which when set to its lowest value, obtains the ordinary erosion and dilation. The relations between the regulated erosion and dilation are studied, and it is shown that these operations may be obtained from each other. The relations between the regulated operations and some other non-linear operations, such as order-statistic filters and soft morphological operations, are described, and it is shown that the regulated operations may have a thresholded linear filtering interpretation.

Based on the basic regulated operations, an anti-extensive regulated erosion and an extensive regulated dilation are defined, and their properties are studied. These operations possess the strictness property of the regulated erosion and dilation operations. Based on the anti-extensive regulated erosion and the extensive regulated dilation, regulated open and close operations are defined, by extending the fitting interpretation of the ordinary open and close. The properties of the regulated open and close are described, and it is shown that they are idempotent. Since the regulated morphological operations possess many of the properties of the ordinary morphological operations, given an existing morphological algorithm, it is possible to try and improve the results obtained by it by using the regulated operations instead of the ordinary operations with strictness that may be optimized according to some optimization criteria. The paper contains many examples of the proposed approach, that demonstrate the advantages obtained by using the regulated operations. An efficient implementation of the regulated morphological operations, which is based on directional interval coding, is described in Ref. [14].

Finally, it should be noted that even though this paper discusses binary morphological operations, it is possible to extend the proposed binary operations to gray-scale operations, by using the fact that the proposed binary operations are increasing with respect to the first argument. By using this fact, it is possible to compose the gray-scale equivalents of the binary operations by processing thresholded sections of the gray-scale image by the binary operations, and then stacking the processed sections in order to obtain the gray-scale result [15].

References

[1] D. Sinha. E.R. Dougherty, Fuzzy mathematical morphology, J. Visual Commun. Image Representation 3 (3) (1992) 286–302.

[2] J. Bloch, H. Maitre, Fuzzy mathematical morphology, Ann. Math. Artificial Intell. 10 (1994) 55–84.

[3] P. Kuosmanen, J. Astola, Soft morphological filtering, J. Math. Imaging Vision 5 (1995) 231–262.

[4] F.Y. Shih, C.C. Pu, Analysis of the properties of soft morphological filtering using threshold decomposition, IEEE Trans. Signal Process. 43 (2) (1995) 539–544.

[5] J. Pitas, A.N. Venetsanopoulos, Order statistics in digital image processing, Proc. IEEE 80 (12) (1992) 1893–1921.

[6] P. Maragos, R.W. Schafer, Morphological filters – part II: Their relations to median, order-statistic, and stack filters, IEEE Trans. ASSP 35 (8) (1987) 1170–1184.

[7] G. Agam, I. Dinstein, Generalized morphological operators applied to map-analysis, in: P. Perner, P. Wang, A. Rosenfeld (Eds), Advances in Structural and Syntactical Pattern Recognition, vol. 1121, Springer, New York, 1996, 60–69.

[8] M.A. Zmuda, L.A. Tamburino, Efficient algorithms for the soft morphological operators, IEEE Trans. PAMI 18 (11) (1996) 1142–1147.

[9] R.M. Haralick, S.R. Sternberg, X. Zhuang, Image analysis using mathematical morphology, IEEE Trans. PAMI 9 (4) (1987) 532–550.

[10] G. Agam, H. Luo, I. Dinstein, Morphological approach for dashed lines detection", in: R. Kasturi, K. Tombre (Eds.), Graphics Recognition: Methods and Applications, vol. 1072 (1996) 92–105.

[11] G. Agam, I. Dinstein, Directional decomposition of line-drawing images based on regulated morphological operations, in: K. Tombre, A.K. Chhabra (Eds.), Graphics Recognition: Algorithms and Systems, vol. 1389, Springer, New York, 1998, pp. 21–34.

[12] G. Agam, I. Dinstein, Adaptive directional morphology with application to document analysis, in: P. Maragos, R.W. Schafer, M.A. Butt (Eds.), Mathematical Morphology and its Applications to Image and Signal Processing, Kluwer Academic Publishers, Dordrecht, 1996, pp. 401–408.

[13] J. Serra, Image Analysis and Mathematical Morphology, Academic Press, New York, 1982.

[14] G. Agam, I. Dinstein, Efficient implementation of regulated morphological operations based on directional interval coding, Proc. SSPR'98, Sydney, Australia, August 1998.

[15] P.D. Wendt, E.J. Coyle, N.C. Gallagher Jr., Stack filters, IEEE Trans. ASSP 34 (4) (1986) 898–911.

About the Author—GADY AGAM received his B.Sc., M.Sc., and Ph.D. degrees in Electrical and Computer Engineering from Ben-Gurion University, Israel, in 1990, 1994, and 1998, respectively. From 1990 to 1991, he was with the electronics division of The Israel Aircraft Industry, Israel. He is currently a post-doctoral fellow in The Department of Electrical and Computer Engineering at

Ecole Polytechnique, Montreal, Canada, working on visual guidance within the context of mining. His other research interests include computational vision, image processing, medical imaging, and document analysis. He is a member of the IEEE.

About the Author—ITS'HAK DINSTEIN received his B.Sc. in Electrical Engineering from The Technion, Israel Institute of Technology, in 1965, and his M.Sc. and Ph.D. degrees in Electrical Engineering from The University of Kansas in 1969 and 1974, respectively. He was with Comsat Labs, Clarksburg, MD, USA, during 1974–1977 working on digital television. He has been with Ben Gurion University, Beer Sheva, Israel, since 1977, where he is now a professor of Electrical Engineering. He was a visiting scientist at IBM Research Laboratory in San-Jose, CA, during 1982–1984, and a visiting associate professor at Polytechnic University, Brooklyn, New-York, during 1988–1990. His current research interests include document understanding and adaptive enhancement and restoration of image sequences. He is an associate editor of The IEEE Transactions of Pattern Analysis and Machine Intelligence, and he is the general Manager of Impro-Temed Ltd.